Lecture notes on abstract harmonic analysis

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This lecture introduces to the basics of unitary representations of topological groups. While classical harmonic analysis studies functions on abelian topological groups, abstract harmonic analysis is concerned with unitary representations of locally compact groups by (functional) analytic means.

1 Topological groups

The very first aim of the lecture is to understand all terminology involved in the announcement. Let us recall the definition of a group.

Definition 1.0.1. A group is a set G with (i) associative multiplication $G \times G \to G : (g, h) \mapsto g \cdot h$, (ii) a neutral element $e \in G$ and (iii) an inversion $G \to G : g \mapsto g^{-1}$. satisfying the commonly known axioms.

Example 1.0.2. Examples of groups that every mathematician must have seen before include:

- S_n, A_n the symmetric and the alternating groups.
- $\mathbb{R}$ the real line.
- S^1 the circle group.
- $\mathcal{O}(n), \mathcal{U}(n)$ the orthogonal and the unitary groups.
- $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ the general linear groups.

While S_n and A_n are finite groups and can be studied as such, it is advantageous to take the natural topology of all the remain examples into account.

Definition 1.0.3 (Topological groups). A topological group is a group G with a topology on G such that multiplication and inversion are continuous maps.

Considering groups as topological groups when possible is not only natural, but will also provide us with additional tools to study them. Finally, we can better control problems of logical kind, by restricting our considerations to separable groups or second countable groups.

Let us first mention that regardless of the use of additional information provided by a topology, every group can be turned into a topological group in a trivial fashion.

Example 1.0.4. Every group is a topological group with the discrete and with the indiscrete topology.

Exercise 1.0.5. Let us consider the previously mentioned examples of groups. While S_n, A_n are naturally discrete, all other examples are instructive.

- $\mathbb{R}$ and $\mathbb{C}$ become topological groups with their usual topology – thanks to the triangle inequality.

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- S^1 inherits a topology from the embedding $S^1 \subset \mathbb{C}$ as vectors of length 1.
- $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ inherit a topology from the embedding $\mathrm{GL}(n, \mathbb{K}) \subset \mathrm{M}_n(\mathbb{K}) \subset \mathbb{K}^{n^2}$.
- $\mathcal{O}(n)$, $\mathcal{U}(n)$ carry the topology of pointwise convergence.

One might notice that, since $\mathcal{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$ and $\mathcal{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$, we actually have two natural topologies on $\mathcal{O}(n)$ and $\mathcal{U}(n)$. It is not difficult however, to show that they agree. This is a reflection of a more general phenomenon, which we describe for the unitary group $\mathcal{U}(H)$ of a (complex) Hilbert space. Before doing so, let us recall one useful point of view on topology.

Recall 1.0.6 (Nets). A directed set I is a partial order such that for all $i, j \in I$ there is $k \in I$ satisfying $k \geq i$ and $k \geq j$. A net in a topological space X is a map $I \to X$ from a directed set into X. We note a net by $(x_i)_{i \in I}$. Such a net converges towards $x \in X$ if for every neighbourhood U of x there is $i_o \in I$ such that $x_i \in U$ for all $i \geq i_0$. The following proposition says that a topological space is completely described by knowing its convergent nets.

Proposition 1.0.7. *Let X be a topological space and $A \subset X$. Then*

$$\overline{A} = \{x \in X \mid \exists (x_i)_i, x_i \in A \text{ such that } x_i \to x\}.$$

Without proof.

Proposition 1.0.8. *Let H be a (complex) Hilbert space. Then the topology of pointwise convergence on $\mathcal{U}(H)$ and the topology of pointwise weak convergences on $\mathcal{U}(H)$ agree.*

Proof. Let $(u_i)_i$ be a net in $\mathcal{U}(H)$ and $u \in \mathcal{U}(H)$. We show that $u_i \to u$ pointwise if and only if $u_i \to u$ pointwise weakly. Since convergence in H implies weak converges in H, the forward implication is clear. So let us assume that $u_i \to u$ pointwise weakly and let $\xi \in H$. Then

$$\|u\xi - u_i\xi\|^2 = \|u\xi\|^2 - 2\,\mathrm{Re}\langle u\xi, u_i\xi\rangle + \|u_i\xi\|^2 = 2\|\xi\|^2 - 2\,\mathrm{Re}\langle u\xi, u_i\xi\rangle \to 0.$$

This shows that $u_i \to u$ pointwise. □

Notation 1.0.9. If H denotes a Hilbert space, we call the topology of pointwise converges on $\mathcal{U}(H)$ the *strong topology*.

Let us add two more examples of topological groups.

Exercise 1.0.10 (Permutation groups). Let X be some set and denote by $\mathrm{Sym}(X)$ the group of all bijections of X. Then $\mathrm{Sym}(X)$ is called the permutation group of X. Check that the following two topologies on $\mathrm{Sym}(X)$ agree and turn it into a topological group.

- The topology of pointwise convergence.
- The coarsest topology on $\mathrm{Sym}(X)$ which is a group topology and turns every point stabiliser $\mathrm{Sym}(X)_x = \{g \in \mathrm{Sym}(X) \mid gx = x\}$ into an open subgroup.

Hint. Denote by τ_{pw} the topology of pointwise convergence on $\mathrm{Sym}(X)$ and by τ_{stab} the coarsest topology on $\mathrm{Sym}(X)$ which is a group topology and turns every point stabiliser $\mathrm{Sym}(X)_x$, $x \in X$ into an open subgroup. First show that τ_{pw} is a group topology. Then note that $\mathrm{Sym}(X)_x \in \tau_{\mathrm{pw}}$ for all $x \in X$. Together, these two facts imply that $\tau_{\mathrm{pw}} \supset \tau_{\mathrm{stab}}$. It remains to show the converse inclusion. To this end, note that a basis of τ_{stab} is given by the sets

$$U_{x,y} = \{g \in \mathrm{Sym}(X) \mid gx = y\} \qquad x, y \in X .$$

For fixed $x, y \in X$ pick some $g \in \mathrm{Sym}(X)$ satisfying $gx = y$. Then $U_{x,y} = g\,\mathrm{Sym}(X)_x$. Since τ_{stab} is a group topology, the map $G \ni h \mapsto gh \in G$ is a homeomorphism of $(\mathrm{Sym}(X), \tau_{\mathrm{stab}})$. So $\mathrm{Sym}(X)_x \in \tau_{\mathrm{stab}}$ implies $g\,\mathrm{Sym}(X)_x \in \tau_{\mathrm{stab}}$. This proves $\tau_{\mathrm{stab}} \supset \tau_{\mathrm{pw}}$. ☺

Example 1.0.11 (Profinite groups). If $(F_i, \varphi_i)_{i \in I}$ is a projective system of finite groups, then the profinite limit $K = \varprojlim (F_i, \varphi_i)$ carries the profinite topology, which is the coarsest topology making all all natural projections $K \to F_i$ continuous. The profinite topology is a group topology and it is compact. For example, (i) Galois groups of field extensions are of this kind. Their topology is referred to as the Krull topology. (Actually all profinite groups appear like this). (ii) Every residually finite group admits a profinite completion, into which it injects as a dense subgroup.

1.1 A characterisation of topological groups

After having described some basic examples of topological groups, let us turn to a characterisations of theirs, which requires a single map to be continuous. This characterisation can be useful to smoothen arguments and to avoid technical repetitions in proofs involving topological groups.

Proposition 1.1.1. *A group G with topology is a topological group if and only if the map $G \times G \to G : (g, h) \mapsto gh^{-1}$ is continuous.*

Proof. Assume that G is a topological group. Then $(g, h) \mapsto gh^{-1}$ is a composition of the continuous maps $(g, h) \mapsto (g, h^{-1})$ and $(g, h) \mapsto gh$, so it is continuous.

Assuming that $(g, h) \mapsto gh^{-1}$ is continuous, we see that $g^{-1} = eg^{-1}$ and hence the inversion is a composition of the two continuous maps $g \mapsto (e, g)$ and $(g, h) \mapsto gh^{-1}$. It follows that also multiplication is a composition of continuous maps $(g, h) \mapsto (g, h^{-1})$ and $(g, h) \mapsto gh^{-1}$. So G is a topological group. □

1.2 Constructions of topological groups

In order to extend our zoo of examples, we note in the next proposition that being a topological groups is preserved under a number of natural constructions available for groups and topological spaces. Its proof is more lengthy than its content deserves, since there are a few traps from elementary topology to avoid. It uses Proposition 1.1.1 freely.

Proposition 1.2.1.

(i) Every subgroup of a topological group is a topological group.

(ii) The product of topological groups is a topological group.

(iii) Quotients of topological groups are topological groups.

Proof. On (i). Let $H \leq G$ be a subgroup of a topological group. Then $H \times H \to H : (h_1, h_2) \mapsto (h_1 h_2^{-1})$ is a restriction of the continuous map $G \times G \to G : (g_1, g_2) \mapsto G$, since the subspace topology on $H \times H \subset G \times G$ agrees with the product topology of $H \times H$. So $H \times H$ is a topological group.

On (ii). Let $(G_i)_{i\in I}$ be a family of topological groups. In order to show that the map $\prod_{i\in I} G_i \times \prod_{i\in I} G_i \to \prod_{i\in I} G_i : ((g_i), (h_i)) \mapsto (g_i h_i^{-1})$ is continuous, it suffices to check that its composition with the quotient maps $\prod_{i\in I} G_i \to G_{i_0}$, $i_0 \in I$, remains continuous. So pick some $i_0 \in I$. Then the following diagram of maps between sets commutes.

$$\begin{tikzcd}
\prod_{i\in I} G_i \times \prod_{i\in I} G_i \arrow[r] \arrow[d] & \prod_{i\in I} G_i \arrow[d] \\
G_{i_0} \times G_{i_0} \arrow[r] & G_{i_0} .
\end{tikzcd}$$

Here the bottom row denotes the map $G_{i_0} \times G_{i_0} \to G_{i_0} : (g, h) \mapsto gh^{-1}$. This shows continuity of

$$\left((\prod_{i\in I} G_i \times \prod_{i\in I} G_i) \to \prod_{i\in I} G_i \to G_{i_0} \right) = \left((\prod_{i\in I} G_i \times \prod_{i\in I} G_i) \to G_{i_0} \times G_{i_0} \to G_{i_0} \right),$$

since the latter is a composition of continuous maps. This shows that $\prod_{i\in I} G_i$ is a topological group.

On (iii). Let $N \trianglelefteq G$ be a normal subgroup of a topological group. We first show that the quotient map $\pi : G \hookrightarrow G/N$ is open. If $U \subset G$ is open, then $\pi^{-1}(\pi(U)) = \bigcup_{g\in N} Ug$ is open in G. Hence, $\pi(U)$ is open in G/N. So it follows that there is a natural homeomorphism between $G/N \times G/N$ with its product topology and $(G \times G)/(N \times N)$ with its quotient topology. We deduce that in order to check continuity of $G/N \times G/N \to G/N : (gN, hN) \mapsto gh^{-1}N$, it suffices to check continuity of its precomposition with the quotient map $G \times G \to G/N \times G/N$. Consider the following commutative diagram.

$$\begin{tikzcd}
G \times G \arrow[r] \arrow[d] & G \arrow[d] \\
G/N \times G/N \arrow[r] & G/N ,
\end{tikzcd}$$

where the top row denotes the map $G \times G \to G : (g, h) \to gh^{-1}$. This shows that

$$(G \times G \to G/N \times G/N \to G/N) = (G \times G \to G \to G/N)$$

is a composition of continuous maps. This shows that G/N is a topological group. ☐

1.3 Homomorphisms of topological groups

After having defined topological groups, we obviously are interested in the correct notion of morhpisms between them.

Definition 1.3.1 (Homomorphisms and isomorphisms). A homomorphism $G \to H$ between topological groups G, H is a continuous group homomorphism $\varphi : G \to H$. An isomorphism $G \to H$ is a bijective homomorphism whose inverse is a homomorphism of topological groups.

Exercise 1.3.2. Construct a non-trivial group homomorphism $\mathbb{R} \to \mathbb{Q}$. Show that this homomorphism considered as a map $\mathbb{R} \to \mathbb{R}$ cannot be continuous.

Hint. Use a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ in order to construct a non-trivial $\mathbb{Q}$-linear functional on $\mathbb{R}$. Then note that continuous maps send connected spaces to connected spaces. ☺

Let us give a useful tool allowing to check more easily whether a given group homomorphisms is a homomorphism of topological groups.

Definition 1.3.3. Let $\varphi : X \to Y$ be a map between topological spaces. We say that φ is continuous at a point $x \in X$ if for every net $x_i \to x$ we have $\varphi(x_i) \to \varphi(x)$.

Proposition 1.3.4. *Let G, H be topological groups. A group homomorphism $G \to H$ is continuous if and only if it is continuous at $e \in G$.*

Proof. Let $\varphi : G \to H$ be a group homomorphism. We only need to show that continuity of φ at e implies continuity of φ. To this end let $g_i \to g$ be a convergent net in G. Then $g_i g^{-1} \to gg^{-1} = e$. So $\varphi(g_i g^{-1}) \to \varphi(e) = e$. This implies $\varphi(g_i) \to \varphi(g^{-1})^{-1} = \varphi(g)$. So φ is continuous. □

1.4 Some technical observations on topological groups

We end this section by pointing out some useful technical statements that hold true in every topological group.

Lemma 1.4.1. *Let G be a topological group. The following statements hold true.*

- *Every neighbourhood of the identity contains a symmetric neighbourhood of the identity.*
- *Every neighbourhood $U \subset G$ of the identity contains a neighbourhood V of the identity such that $V^2 \subset U$.*
- *If $A, B \subset G$ are compact, then $AB \subset G$ is compact.*
- *If $A \subset G$ is compact and $B \subset G$ is closed, then $AB \subset G$ and BA are closed.*
- *If $A \subset G$ is open and $B \subset G$ is an arbitrary subset, then AB and BA are open.*

Proof. If $O \ni e$ is open, then $O \cap O^{-1}$ is a symmetric open set containing e. In particular, every neighbourhood of the identity contains a symmetric neighbourhood of the identity.

Let $U \subset G$ be a neighbourhood of e. We may assume that U is open. Then the inverse image of U under the multiplication map $\mathrm{m} : G \times G \to G : (g, h) \mapsto gh$ is open and contains (e, e). By the definition of the product topology there are open sets $V_1, V_2 \ni e$ such that $V_1, V_2 \subset U$. Putting $V := V_1 \cap V_2$, we found an open set $V \ni e$ satisfying

$$V^2 \subset V_1 V_2 \subset U\,.$$

Let $A, B \subset G$ be compact subsets. Then $AB = \mathrm{m}(A \times B)$ is the image of a compact set under a continuous map. Hence AB is compact.

Let $A \subset G$ be compact and $B \subset G$ be closed. We show that AB is closed. To this end let (g_i) be a convergent net with $g_i \in AB$. There are nets (a_i) and (b_i) in A and B, respectively, such that $a_i b_i = g_i$ for all i. Since A is compact, we may pass to a subnet of (g_i) and assume that (a_i) converges to some $a \in A$. Then $b_i = a_i^{-1} g_i \to a^{-1}g$, which lies in the closed set B. So $g_i \to a(a^{-1}g) \in AB$. This shows that AB is closed. Similarly, it follows that BA is closed.

If $A \subset G$ is open and B is arbitrary, then $AB = \bigcup_{g \in B} Ab$ and $BA = \bigcup_{g \in G} bA$ are unions of open sets and hence open. □

2 Hausdorff groups

In this section we will investigate when the underlying space of a topological group is Hausdorff. It turns out that this is automtic under very mild assumptions. While non-Hausdorff groups can be interesting in principal (think of algebraic groups with their Zariski topology for example), doing any kind of analysis on topological groups automatically puts us into the Hausdorff setting, as we will see.

We start by recall two separation axioms from elementary topology.

Recall 2.0.1. Let X be a topological space.

- X is called a *Kolmogorov space* or a T_0 *space* if for every two distinct points $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$.
- X is called a T_1*-space* if for every two distinct points $x, y \in X$ there is an open set $U \subset X$ such that $x \in U$ and $y \notin U$. Equivalently, every point in X is closed.
- X is called a *Hausdorff space* if for every two distinct points $x, y \in X$ there are disjoint open set $U, V \subset X$ such that $x \in U$, $y \in V$.

Let us formally fix the notion of a Hausdorff group.

Definition 2.0.2. A *Hausdorff group* is a topological group whose underlying topology is Hausdorff.

2.1 T_0-groups are Hausdorff

We will show that every topological group whose topology is T_0 already follows Hausdorff. This establishes a big gap between being Hausdorff or not for topological groups. We start by giving a short characterisation of Hausdorff spaces in terms of homomorphisms.

Proposition 2.1.1. *Let X be a topological space. Then X is Hausdorff if and only if the diagonal in $X \times X$ is closed.*

Proof. Assume that the diagonal $\Delta \subset X \times X$ is closed and let $x, y \in X$ be distinct points. There is a basic open $U \times V \subset X \times X \setminus \Delta$ which contains (x, y). Then $U \ni x$, $V \ni y$ and $U \cap V = \emptyset$, since $U \times V \cap \Delta = \emptyset$.

Now assume that X is Hausdorff. We show that $X \times X \setminus \Delta$ is open. To this end let $(x, y) \in X \times X \setminus \Delta$. There are disjoint open sets $U, V \subset X$ such that $U \ni x$, $V \ni y$. Then $X \times X \setminus \Delta \supset U \times V \ni (x, y)$. □

Proposition 2.1.2. *Let G be a topological group. Then the following statements are equivalent.*

- *G is a T_0-space.*
- *$\{e\} \subset G$ is closed.*
- *G is a T_1-space.*
- *G is a Hausdorff space.*

Proof. Assume that G is a T_0-space. We show that $\{e\}$ is closed in G. To this end, it suffices to exhibit for every $g \in G \setminus \{e\}$ some open set $U \subset G$ such that $g \in U$ but $e \notin U$. Take $g \in G$. Since G is a T_0-space there is some open set $U \subset G$ such that either $g \in U$, $e \notin U$ or $e \in U$, $g \notin U$. In the former case we are done. In the latter case consider the set $U^{-1}g$. Then $g \in U^{-1}g$, since $e \in U^{-1}$. Further, we have the equivalence $e \in U^{-1}g \Leftrightarrow g^{-1} \in U^{-1} \Leftrightarrow g \in U$, which shows that $e \notin U^{-1}g$. It follows that $\{e\}$ is closed in G.

If $\{e\} \subset G$ is closed, then $g\{e\} = \{g\}$ is closed for all $g \in G$. Hence G is a T_1-space.

Assume that G is a T_1 space. Consider the inverse image of $\{e\}$ under the map $G \times G \ni (g,h) \mapsto gh^{-1} \in G$, which must be closed. This is the diagonal of G, since $gh^{-1} = e$ if and only if $g = h$. So the diagonal of $G \times G$ is closed. Now Proposition 2.1.1 applies and shows that G is a Hausdorff space.

Finally, every Hausdorff space is a T_0-space, which finishes the proof of the proposition. □

2.2 The Kolmogorov quotient and the Hausdorff quotient of a topological group

We show that every topological group has a maximal Hausdorff quotient and that every continuous map (not necessarily a group homomophism) factors through this quotient. This gives strong support to the idea that any study of topological groups by analytic means must restrict considerations to Hausdorff groups.

We start by introducing the Kolmogorov quotient of a topological space.

Definition 2.2.1. Let X be a topological space. Two points $x, y \in X$ are called *topologically indistinguishable*, if for every open subset $U \subset X$, we have $x \in U \Leftrightarrow y \in U$.

Proposition 2.2.2. *Let X be a topological space. Then topological indistinguishability is an equivalence relation on X, which we denote by $\sim$. The quotient $X/\sim$ is a T_0-space.*

Proof. One checks right away that topological indistinguishably is is an equivalence relation on X. We show that $X/\sim$ is a T_0-space. Let $\pi : X \to X/\sim$ be the quotient map and $U \subset X$ open and $x \in X$. If $x \in \pi(U)$ then there is $y \in U$ such that $x \sim y$. But this implies $x \in U$, since U is open. We showed $\pi^{-1}(\pi(U)) = U$.

If now $[x] \neq [y]$ are different points in $X/\sim$, then x and y are topologically distinguishable in X. So there is an open subset $U \subset X$ such that either $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$. Renaming x and y, we may assume that the former is the case. Since $\pi^{-1}(\pi(U)) = U$, it follows that $\pi(U) \subset X/\sim$ is open and $[y] \notin \pi(U)$. So $[x], [y]$ are topologically distinguishable in $X/\sim$. This shows that $X/\sim$ is a T_0-space. □

The quotient $X \to X/\sim$ from the previous proposition is called the *Kolmogorov quotient* of X. It actually enjoys the following universal property.

Proposition 2.2.3. *Let X be a topological space and $\pi : X \to Y$ the Kolmogorov quotient of X. Then every continuous map $f : X \to Z$ into a T_0-space factors through π.*

Proof. We claim that topologically indistinguishable points in X have the same image under f. To prove this, assume that $x, y \in X$ satisfy $f(x) \neq f(y)$. Then there is an open set $U \subset Z$ such that either $f(x) \in U$, $f(y) \notin U$ or $f(y) \in U$, $f(x) \notin U$. Renaming x and y we may assume that the former is the case. Let $O := f^{-1}(U)$, which is an open subset of X. Then $x \in O$ but $y \notin O$, showing that x, y are topologically distinguishable in X. This proves our claim and shows that f factors through the Kolmogorov quotient. □

We will now show that the Kolmogorov quotient of a topological group is a quotient onto a Hausdorff group. This quotient is universal.

Proposition 2.2.4. *Let $H \leq G$ be a subgroup of a topological group G. Then the closure $\overline{H}$ is a subgroup of G. If $H \trianglelefteq G$ is normal, then $\overline{H} \trianglelefteq G$ is normal too.*

Proof. This is elementary. □

Exercise 2.2.5. Let G be a topological group and $N \trianglelefteq G$ a normal subgroup. Show that the quotient G/N is a Hausdorff group if and only if $N \leq G$ is closed.

Hint. Use the fact that G/N is a Hausdorff group if and only if $\{N\} \subset G/N$ is closed. ☺

Proposition 2.2.6. *Let G be a topological group and $N := \overline{\{e\}} \trianglelefteq G$. Then G/N is a Hausdorff group and $G \to G/N$ is the Kolmogorov quotient of G.*

Proof. By Proposition 2.2.4, we know that $N \trianglelefteq G$ is a closed normal subgroup, so that G/N is a Hausdorff group by Exercise 2.2.5. Denote by $\sim$ the equivalence relation of being topologically indistinghuishable in G. In order to show that $G \to G/N$ is the Kolmogorov quotient of G, we have to show that $g \sim h \Leftrightarrow gh^{-1} \in N$. Since $g \sim h \Leftrightarrow gh^{-1} \sim e$, this is equivalent to showing $N = [e]$, where $[e]$ denotes the equivalence class of e with respect to $\sim$.

Since $N \subset G$ is closed and contains e, the inclusion $[e] \subset N$ follows. We prove the converse inclusion. If $g \sim e \sim h$, then $gh^{-1} \sim e$ follows, which shows that $[e]$ is a subgroup of G. Further, $g[e] = [g] = [e]g$ for all $g \in G$ implying that $[e]$ is normal in G. The quotient $G/[e]$ is the Kolmogorov quotient of G, since $[g] = [h] \Leftrightarrow [gh^{-1}] = [e]$. In particular, $G/[e]$ is a T_0-space and hence Hausdorff by Proposition 2.1.2. So $[e] \subset G$ is closed subgroup by Exercise 2.2.5. Since $e \in [e]$, this implies $N \subset [e]$, which was to be shown. □

Definition 2.2.7. Let G be a topological group and $N := \overline{\{e\}}$. Then G/N is called the *Hausdorff quotient* of G.

The universality of the Kolmogorov quotient (Proposition 2.2.3) shows now the following universal property of the Hausdorff quotient of a topological group. This corollary justifies to restrict our considerations to Hausdorff groups in all subsequent sections.

Corollary 2.2.8. *Let G be a topological group and $N := \overline{\{e\}} \trianglelefteq G$. Then every continuous map from G into a T_0-space (such as $\mathbb{R}$ or $\mathbb{C}$) factors through $G \to G/N$.*

3 Unitary representations

Historically, the concept of a group arose in mathematics through the study of concrete groups of symmetries. However, the modern point of view has changed perspectives. We nowadays fix a group and then study all possible ways it can arise as group of symmetries of some class of objects. We will restrict our attention first to topological vector spaces, then to Banach spaces and finally to Hilbert spaces.

Definition 3.0.1 (Representation). G be a topological group and V a topological vector space. We denote by $\mathrm{GL}(V)$ the group of all continuous vector space isomorphisms with a continuous inverse. A representation of G on V is a homomorphism $G \to \mathrm{GL}(V)$ such that the map $G \times V \to V$ is continuous.

Remark 3.0.2. Note that the definition of a representation of G on V asks more then merely a homomorphism of topological groups $G \to \mathrm{GL}(V)$, for $\mathrm{GL}(V)$ carrying the topology of pointwise convergence. This is equivalent to asking the map $G \times V \to V$ to be separately continuous, which is weaker than our assumption of (joint) continuity.

Example 3.0.3. Every topological group admits the trivial representation $G \to \mathcal{U}(1)$ mapping all elements to the identity on $\mathbb{C}$. Here are some other natural representations of certain classes of groups.

- The group $\mathrm{Sym}(X)$ admits the permutation representation on $\ell^2(X)$ permuting elements from the natural orthonormal basis.
- The general linear groups are represented by the identical map $\mathrm{GL}(n, \mathbb{C}) \curvearrowright \mathbb{C}^n$.
- Also the unitary groups are represented by their inclusion $\mathcal{U}(n) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$.
- A character of an abelian group A is a homomorphism $A \to \mathrm{S}^1$. Every character corresponds to a unitary representation by the natural identification $\mathrm{S}^1 \cong \mathcal{U}(1)$.

Turning our attention to representations on a Banach space V for a moment, we obtain a useful criterion to check whether a given homomorphism $G \to \mathrm{GL}(V)$ is a representation of not.

Proposition 3.0.4. *Let G be a topological group and V a Banach space. An (abstract) homomorphism $\pi : G \to \mathrm{GL}(V)$ is a representation if and only if it is continuous and the map $G \to \mathbb{R}_{>0} : g \mapsto \|\pi(g)\|$ is locally bounded.*

Proof. Assume that $\pi : G \to \mathrm{GL}(V)$ is a representation. We first show continuity of π. Let $g_i \to g$ be a convergent net in G and let $v \in V$. Then $\pi(g_i)v \to \pi(g)v$, since $G \times V \to V$ in particular is continuous in the first variable. So π is continuous. By continuity of $G \times V \to V$, there is an open neighbourhood U of $e \in G$ and an open neighbourhood O of $0 \in V$ such that $\pi(U)O \subset \mathrm{B}(0, 1)$. Since O is open, there is some $r > 0$ such that $\mathrm{B}(0, r) \subset O$. This means that $\|\pi(g)\| < r^{-1}$ for all $g \in U$. It follows that $g \mapsto \|\pi(g)\|$ is bounded on all sets of the form hU, $h \in G$, which proves local boundedness of $g \mapsto \|\pi(g)\|$.

Assume now that $G \to \mathrm{GL}(V)$ is a continuous homomorphism such that $g \mapsto \|\pi(g)\|$ is locally bounded. Since $G \to \mathrm{GL}(V)$ is continuous, the map $g \mapsto \pi(g)v$ is continuous for all $v \in V$. We then obtain for $g, h \in G$ and $v, w \in V$

$$\|\pi(g)v - \pi(h)w\| \leq \|\pi(h)\| \|\pi(h^{-1}g)v - w\| \leq \|\pi(h)\| (\|\pi(h^{-1}g)v - v\| + \|v - w\|) ,$$

which is small if h and w are fixed and g is close to h and v is close to w. This shows continuity of $G \times V \to V$ and hence that π is a representation. $\square$

We are next going to restrict our attention to representations on Hilbert spaces. This has two reasons. First, from the analytic point of few, the theory of unitary representations in much better tractable than other classes of representations. Second, unitary representations are the natural objects of interest in physics, since symmetries must preserve measurements, the latter being modelled by scalar products in Hilbert space.

Definition 3.0.5 (Unitary representation). Let G be a topological group and H a complex Hilbert space. A unitary representation of G on H is a homomorphism (of topological groups) $G \to \mathcal{U}(H)$. We write (π, H) for this unitary representation. A representation $\pi : G \to \mathrm{GL}(V)$ is called unitarisable if there is a compatible scalar product on V with respect to which π is unitary.

Exercise 3.0.6. Show that the representation $\mathrm{id} : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is not unitarisable.

Hint. Check that if $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathbb{C}^n$, then $\alpha \in \mathbb{C}\mathrm{id} \subset \mathrm{GL}(n, \mathbb{C})$ does not act as a unitary unless $|\alpha| = 1$. ☺

Exercise 3.0.7. Show that every representation of a finite group on a Hilbert space is unitarisable.

Hint. Consider $F \to \mathrm{GL}(H)$ be a representation of the finite group F on the Hilbert space H. Averaging the scalar product of H over the action of F, obtain a new scalar product on H witnessing unitarisability of $F \to \mathrm{GL}(H)$. ☺

Remark 3.0.8. There are groups without any non-trivial unitary representations. For example the group $\mathrm{H}^+([0, 1])$ of orientation preserving homeomorphisms of the interval is of this kind. It is however non-trivial to prove this.

Definition 3.0.9 (Unitary equivalence of representations). Two unitary representations $(\pi_1, H_1), (\pi_2, H_2)$ of a topological group G are called unitarily equivalent, if there is a unitary operator $U : H_1 \to H_2$ such that $U\pi_1(g) = \pi_2(g)U$ for all $g \in G$. We write in this case $\pi_1 \cong \pi_2$.

3.1 Constructions of unitary representations

Similar to the list of construction for topological groups in Section 1.2, we introduce a number of important constructions for unitary representations.

Definition 3.1.1 (Direct sum of unitary representations). Let ν, π be unitary representations of a topological group G on Hilbert spaces H_ν and H_π. The direct sum $\nu \oplus \pi$ is the unitary representation $G \mapsto \mathcal{U}(H_\nu \oplus H_\pi)$ given by $(\nu \oplus \pi)(g)(\xi \oplus \eta) = \nu(g)\xi \oplus \pi(g)\eta$. Similarly the direct sum of any family of representations can be defined.

The direct sum construction allows us to make the following observation about having "sufficently many" unitary representations. We say that a topological group G has *sufficiently many unitary representations* if for every $g \in G$ there is a unitary representation $\pi : G \to \mathcal{U}(H)$ such that $\pi(g) \neq \mathrm{id}_H$.

Proposition 3.1.2. *A topological group G has sufficiently many unitary representations if and only if there is a Hilbert space H such that $G \hookrightarrow \mathcal{U}(H)$.*

Proof. It is clear that G admits sufficiently many unitary representations if $G \hookrightarrow \mathcal{U}(H)$. Assume that a topological group G admits sufficiently many unitary representations. Let $(\pi_g)_{g \in G}$ be a family of unitary representations such that $\pi_g(g) \neq \mathrm{id}_{H_g}$ for all $g \in G$. The direct sum representation $\bigoplus_{g \in G} \pi_g$ then defines an injective homomorphism $G \hookrightarrow \mathcal{U}(\bigoplus_{g \in G} H_g)$. ☐

Let us next turn to subrepresentations and the decomposition of unitary representations.

Definition 3.1.3 (Subrepresentations). Let π be a unitary representation of a topological group G on a Hilbert space H. A closed subspace $K \leq H$ is π-invariant (or G-invariant) if $\pi(g)K = K$ for all $g \in G$. The induced map $G \to \mathcal{U}(K)$ is a called a *subrepresentation* of π and denoted by $\pi|_K$. If ν is another representation of G that is equivalent to a subrepresentation of π, we write $\nu \leq \pi$.

Proposition 3.1.4. *Let π be a unitary representation of a topological group G on a Hilbert space H. If $K \leq H$ is an invariant subspace, then $K^\perp$ is an invariant subspace. Further, $\pi \cong \pi|_K \oplus \pi|_{K^\perp}$.*

Proof. Let $\xi \in K^\perp$ and $\eta \in K$. Then for all $g \in G$

$$\langle \pi(g)\xi, \eta \rangle = \langle \xi, \pi(g^{-1})\eta \rangle = 0\,.$$

So $\pi(g)\xi \in K^\perp$, showing invariance of $K^\perp$. It now easy to check that the unitary operator $K \oplus K^\perp \to H : (\xi, \eta) \mapsto \xi + \eta$ witnesses $\pi|_K \oplus \pi|_{K^\perp} \cong \pi$. □

The previous proposition allows us to reasonably define the multiplicity of a subrepresentation.

Definition 3.1.5 (Multiplicity). Let π, ρ be unitary representations of a topological group G. Then the multiplicity of ρ in π is the largest cardinal κ such that $\bigoplus_\kappa \rho \leq \pi$.

We next turn to the tensor product construction. It is well-compatible with the concept of unitary representations.

Proposition 3.1.6. *Let ν, π be unitary representations of a topological group G on Hilbert spaces H_ν and H_π. For $g \in G$ denote by $(\nu \otimes \pi)(g) = \nu(g) \otimes \pi(g) \in \mathcal{B}(H_\nu \otimes H_\pi)$ the tensor product of operators. Then $\nu \otimes \pi : G \to \mathcal{U}(H_\nu \otimes H_\pi)$ is continuous.*

Sketch of a proof. Since $(\nu \otimes \pi)(g)$ is unitary for all $g \in G$, it suffices to check continuity on the dense subset $H_\nu \otimes_{\mathrm{alg}} H_\pi$ of $H_\nu \otimes H_\pi$, which is done by an elementary calculation. □

Definition 3.1.7 (Tensor product of unitary representations). Let ν, π be unitary representations of a topological group G on Hilbert spaces H_ν and H_π. The tensor product representation $\nu \otimes \pi$ is given by the map $g \mapsto \nu(g) \otimes \pi(g) \in \mathcal{U}(H_\nu \otimes H_\pi)$.

The last construction we want to consider is the so called conjugate representation. The subsequent proposition justifies thinking of conjugate representations a kind of inverse for the class of (finite dimensional) unitary representations equipped with the tensor product.

Definition 3.1.8 (Conjugate Hilbert space). Let H be a Hilbert space. Then the conjugate Hilbert space of H is $\overline{H} = \{\overline{\xi} \mid \xi \in H\}$ with scalar multiplication $\alpha\overline{\xi} = \overline{\overline{\alpha}\xi}$ and scalar product $\langle \overline{\xi}, \overline{\eta} \rangle = \overline{\langle \xi, \eta \rangle}$ for all $\xi, \eta \in H$, $\alpha \in \mathbb{C}$.

Definition 3.1.9 (Conjugate representation). Let π be a unitary representation of a topological group G on a Hilbert space H. Then $\overline{\pi}(g)\overline{\xi} = \overline{\pi(g)\xi}$ for $\xi \in H$ defines a unitary representation of G called the conjugate representation of π.

Proposition 3.1.10. *Let π be a finite dimensional representation of a topological group G. Then the trivial representation of G is a subrepresentation of $\pi \otimes \overline{\pi}$.*

Proof. It suffices to show that the natural representation of $\mathcal{U}(n)$ on $\mathbb{C}^n$ satisfies the conclusion of the proposition. Let $(e_i)_{1 \leq i \leq n}$ be the standard ONB of $\mathbb{C}^n$ and let $\xi = \sum_i e_i \otimes \overline{e_i} \in \mathbb{C}^n \otimes \overline{\mathbb{C}^n}$. For a unitary $U = (u_{ij})$ we obtain

$$\begin{aligned}(U \otimes \overline{U})\xi &= \sum_i U e_i \otimes \overline{U e_i} \\ &= \sum_{i, j_1, j_2} u_{j_1 i} e_{j_1} \otimes \overline{u_{j_2 i} e_{j_2}} \\ &= \sum_{i, j_1, j_2} u_{j_1 i} \overline{u_{j_2 i}} e_{j_1} \otimes \overline{e_{j_2}} \\ &= \sum_{j_1, j_2} \delta_{j_1, j_2} e_{j_1} \otimes \overline{e_{j_2}} \\ &= \sum_j e_j \otimes \overline{e_j} \\ &= \xi\end{aligned}$$

So ξ is an invariant vector, which finishes the proof. □

3.2 Irreducible representations and the Lemma von Schur

Let us start this section by fixing the notion of irreducible representations.

Definition 3.2.1 (Irreducible unitary representation). A unitary representation is called irreducible, if it admits no non-trivial unitary subrepresentation.

The Lemma von Schur says roughly speaking that a unitary equivalence between two irreducible unitary representations is unique up to scalar multiples. We present an operator algebraic proof, which is the reason for introducing the following terminology.

Definition 3.2.2. Let $\mathcal{S} \subset \mathcal{B}(H)$ be a set of bounded operators on a Hilbert space H.

- $\mathcal{S}$ acts topologically irreducible if the only closed $\mathcal{S}$-invariant subspaces of H are $\{0\}$ and H.
- The commutant of $\mathcal{S}$ is

$$\mathcal{S}' = \{T \in \mathcal{B}(H) \,|\, \forall S \in \mathcal{S} : ST = T\} .$$

Recall 3.2.3. Let us recall some facts about operators on a Hilbert space that we need in the next proposition.

- A set of operators is self-adjoint if it contains all the adjoints of its elements.
- The spectrum of an operator $S \in \mathcal{B}(H)$ is the set of all $\alpha \in \mathbb{C}$ such that $S - \alpha \mathrm{id}_H$ is non-invertible. It is denoted by $\sigma(S)$ and generalises the set of eigenvalues of an operator on a complex vector space.
- The spectral theorem for normal (in particular self-adjoint) operators on a Hilbert space provides us with so called functional calculus. If $S \in \mathcal{B}(H)$ is a normal operator on a Hilbert space, then the unital norm closed *-subalgebra of $\mathcal{B}(H)$ generated by S is isomorphic with $\mathrm{C}(\sigma(S))$. So it makes sense to write $f(S)$ for $f \in \mathrm{C}(\sigma(S))$ and this assignment is multiplicative: $f(S)g(S) = (fg)(S)$ for all $f, g \in \mathrm{C}(\sigma(S))$.

- An operator S on a Hilbert space is a projection if and only if $\sigma(S) \subset \{0,1\}$.

Proposition 3.2.4 (Lemma von Schur, Version for sets of operators). *Let $\mathcal{S} \subset \mathcal{B}(H)$ be a self-adjoint set of operators on a Hilbert space H. Then $\mathcal{S}$ acts topologically irreducible on H if and only if $\mathcal{S}' = \mathbb{C}1$.*

Proof. Assume that $\mathcal{S}' = \mathbb{C}1$ and let $K \leq H$ be a closed $\mathcal{S}$-invariant subspace of H. The orthogonal projection $p : H \to K \subset H$ is an element of $\mathcal{B}(H)$. It satisfies $Sp = pSp$ for all $S \in \mathcal{S}$. Because $\mathcal{S}$ is self-adjoint, it follows that

$$Sp = pSp = (pS^*p)^* = (S^*p)^* = pS\,.$$

So $p \in \mathcal{S}'$. Hence $p \in \{0,1\}$ meaning that $K \in \{\{0\}, H\}$. This means that $\mathcal{S}$ acts topologically irreducible.

Assume now that $\mathcal{S}$ acts topologically irreducible on H. Let $T \in \mathcal{S}'$. Since $\mathcal{S}$ is self-adjoint, $\mathcal{S}'$ is a *-algebra. So we can split T in its real part $\operatorname{Re} T = \frac{1}{2}(T + T^*)$ and its imaginary part $\operatorname{Im} T = \frac{1}{2i}(T - T^*)$ and hence assume $T = T^*$. It then suffices to show that the spectrum $\sigma(T)$ of T is one point. Assuming the contrary, we find non-zero continuous functions, $f, g \in \mathrm{C}(\sigma(T))$ such that $f \cdot g = 0$. Since $\mathcal{S}'$ is a unital norm closed *-subalgebra of $\mathcal{B}(H)$, we have well-defined non-zero elements $f(T), g(T) \in \mathcal{S}'$. By topological transitivity, the non-trivial $\mathcal{S}$-invariant subspace $f(T)H$ must be dense in H. This contradicts $g(T)f(T)H = \{0\}$. So indeed the spectrum of T is one point, hence $T \in \mathbb{C}1$. □

Corollary 3.2.5 (Lemma von Schur, Version for representations). *Let (π, H) be a unitary representation of a group G. Then π is irreducible if and only if every bounded operator $T \in \mathcal{B}(H)$ such that $T\pi(g) = \pi(g)T$ for all $g \in G$ satisfies $T \in \mathbb{C}1$.*

Proof. Since π is unitary, we have $\pi(g)^* = \pi(g^{-1})$. This implies that $\pi(G)$ is a self-adjoint subset of $\mathcal{B}(H)$. Hence $\pi(G)$ acts topologically irreducible if and only if $\pi(G)' = \mathbb{C}1$, by Proposition 3.2.4. Now $\pi(G)$ acts topologically irreducible if and only if π is an irreducible representation. Further, $\pi(G)' = \mathbb{C}1$, if and only if every bounded operator $T \in \mathcal{B}(H)$ satisfying $T\pi(g) = \pi(g)T$ for all $g \in G$ lies in $\mathbb{C}1$. This finishes the proof of the corollary. □

Here is a application of the Lemma von Schur.

Exercise 3.2.6. Let G be an abelian group. Show that irreducible representations of G correspond to its characters under the identification $\mathcal{U}(1) = \mathrm{S}^1$.

Hint. It suffices to show that every irreducible unitary representation of an abelian group is one-dimensional. ☺

We finish this section by introducing the set of all isomorphism classes of irreducible representations of a locally compact group.

Lemma 3.2.7. *Let G be a locally compact group. Then every irreducible unitary representation of G acts on a Hilbert space whose dimension is bounded by the cardinality of G. It follows that the class of isomorphism classes of irreducible unitary representations of G is a set.*

Proof. If (π, H) is an irreducible representation of G and $\xi \in H$ is a non-zero vector, then $G\xi$ is a generating subset of H whose cardinality is bounded by the cardinality of G. The Gram-Schmidt algorithm paired with transfinite induction hence shows that H admits an orthonormal basis whose cardinality is bounded by the cardinality of G. □

Definition 3.2.8. Let G be a locally compact group, then $\hat{G}$ denotes the set of all isomorphism classes of irreducible unitary representations of G.

4 Locally compact groups

In this chapter we will restrict our attention to topological groups whose underlying topology is locally compact. Let us provide three reasons for this restriction. First, considering locally compact groups only, provides us with a so called Haar measure on the group, which is a powerful tool in all questions of analysis. (Section 4.2). Second, locally compact groups admit so the called regular representation. This unitary representation is characterised by the amazing "absorbtion principle" and provides a natural habitat for studying locally compact groups as naturally embedded into the unitary group of a Hilbert space. (Section 4.3). Third, the technical side of our treatment automatically restricts us to locally compact groups, since functional analytic considerations involving the space of compactly supported continuous automatically force us to consider locally compact groups. After all, $\mathrm{C_c}(G) \neq 0$ implies the existence of a relatively compact open subset in G.

Definition 4.0.1 (Locally compact groups). A topological space is called locally compact, if all its points admit a compact neighbourhood. A Hausdorff group whose underlying topology is locally compact, is called a *locally compact group*.

Let us consider which of the previously considered examples of topological groups (Exercises 1.0.5 & 1.0.10 and Examples 1.0.11) are locally compact.

Example 4.0.2.

- All discrete groups are locally compact.
- The the real line $\mathbb{R}$, the complex plane $\mathbb{C}$ and its closed subset S^1 are all locally compact.
- The general linear groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are open subsets of their respective matrix algebras and are hence locally compact.
- The finite dimensional orthogonal and the unitary groups $\mathcal{O}(n)$ and $\mathcal{U}(n)$ are closed subsets of the general linear groups, and hence locally compact.
- Profinite groups are compact and hence they are locally compact.

Let us add some examples of non-locally compact groups.

- $\mathcal{U}(H)$ for an infinite dimensional Hilbert space H however is not locally compact. Indeed, identifying $H \cong \ell^2(\mathbb{Z}) \otimes H$, it suffices to check that the biliteral shift

$$U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) : \delta_n \mapsto \delta_{n+1}$$

  is a unitary whose powers converge to 0 weakly.
- If X is an infinite set, then $\mathrm{Sym}(X)$ is not locally compact. A similar argument as for $\mathcal{U}(H)$, H infinite dimensional, applies.
- $\mathbb{Q} \leq \mathbb{R}$ with the subspace topology is not a locally compact group. Indeed every neighbourhood of $0 \in \mathbb{Q}$ contains a sequence converging to an irrational number in $\mathbb{R}$.

Exercise 4.0.3. We argued that if X is an infinite set, then $\mathrm{Sym}(X)$ is not a locally compact groups. However, $\mathrm{Sym}(X)$ contains a large number of naturally defined locally compact group. If Γ is a graph with edge set $\mathrm{E}(\Gamma)$, then $\mathrm{Aut}(\Gamma) \leq \mathrm{Sym}(\mathrm{E}(\Gamma))$ is closed and a subgroup. Prove that if Γ is locally finite, then $\mathrm{Aut}(\Gamma)$ is locally compact.

Hint. Recall that Γ is locally finite if for every vertex $v \in \mathrm{V}(\Gamma)$ the set of adjacent edges $\{e \in \mathrm{E}(\Gamma) \mid \mathrm{s}(e) = v\}$ is finite. Since the automorphism group of a disjoint union of graphs is the product of the the automorphism groups of its connected components, we can restrict our considerations to the case where $\mathrm{V}(\Gamma)$ is countable. Since Γ is locally finite this implies that $\mathrm{E}(\Gamma)$ is countable and hence $\mathrm{Sym}(\mathrm{E}(\Gamma))$ is separable. It then suffices to check that the stabiliser $\mathrm{Aut}(\Gamma)_v = \{g \in \mathrm{Aut}(\Gamma) \mid gv = v\}$ is sequentially compact for some $v \in \mathrm{V}(\Gamma)$. Using the fact that Γ is locally finite, we can successively find subsequences of $(g_n)_n$ which are constant on balls of large radius around v. A diagonal sequence argument then finishes the proof. ☺

4.1 Uniform continuity

We will need frequently need the following strengthening of the notion of continuity for functions on locally compact groups. Uniform continuity in the sense of the following definition generalises the notion of uniform continuity with respect to some metric. The linking concept is the theory of "uniformities", which will not be explained any further here.

Definition 4.1.1. Let G be a locally compact group and $f : G \to \mathbb{C}$ a function.

- Then f is called left uniformly continuous, if for all $\varepsilon > 0$ there is an open neighbourhood U of the identity in G such that for all $x, y \in G$ satisfying $x \in Uy$ we have $|f(x) - f(y)| < \varepsilon$.
- f is called right uniformly continuous, if for all $\varepsilon > 0$ there is an open neighbourhood U of the identity in G such that for all $x, y \in G$ satisfying $y \in xU$ we have $|f(x) - f(y)| < \varepsilon$.
- f is called uniformly continuous, if it is left- and right uniformly continuous.

Lemma 4.1.2. *Every continuous and compactly supported function on a locally compact group is uniformly continuous.*

Proof. Lemma 1.3.7 in Deitmar-Echterhoff. □

4.2 The Haar measure

Recall 4.2.1. If X is a locally compact space, then an (outer) Radon measure on X is a locally finite measure on the Borel σ-algebra $\Sigma(X)$ of X which satisfies the following approximation properties.

- $\mu(A) = \inf_{\substack{U \supset A \\ U \text{ open}}} \mu(U)$ for every measurable set $A \in \Sigma(X)$
- $\mu(U) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \mu(K)$ for every open subset set $U \subset X$

Note the subtlety that the second approximation property only holds for open subsets of X. In case X is σ-compact (which will be most often the case in applications) this subtlety vanishes and approximation from inside by compact subsets holds for arbitrary measurable sets. See Chapter 1.3 and Appendix B.2 of [DE14] for more details on this problem.

Definition 4.2.2. Let G be a locally compact group and μ an outer Radon measure on G.

- μ is called *left-invariant* or a *left Haar measure* if $\mu(gA) = \mu(A)$ holds for all measurable subsets $A \subset G$ and all $g \in G$.

- μ is called *right-invariant* or a *right Haar measure* if $\mu(gA) = \mu(A)$ holds for all measurable subsets $A \subset G$ and all $g \in G$.

In these notes we will mostly use *left* Haar measures and simply refer to them as Haar measures.

The following two theorems are of crucial importance for the theory of locally compact groups.

Theorem 4.2.3. *Every locally compact group has a Haar measure.*

Theorem 4.2.4. *The Haar measure of a locally compact groups is unique up to a scalar multiple from $\mathbb{R}_{>0}$.*

We will not reproduce the proofs of Theorem 4.2.3 & 4.2.4 here, since they are very well explained in Chapter 1.3 of [DE14].

We will freely use the correspondence between outer Radon measures on G and positive linear functionals on $\mathrm{C_c}(G)$, provided by Riez representation theorem (see [Appendix B.2][DE14]). It allows us to speak of *Haar integrals* $I : f \mapsto \int_g f(x)\mathrm{d}x$ where we integrate against a Haar measure. A Haar integral satisfies the invariance property $I({}^g f) = I(f)$ for all $g \in G$ and all $f \in \mathrm{C_c}(G)$. Up to scalar multiple a Haar integral is uniquely determined by this correspondence.

The uniqueness of the Haar measure allows us to quantify the difference between left and right Haar measure. This is done by means of the next proposition.

Proposition 4.2.5. *Let G be a locally compact group with a left Haar measure μ. For every $g \in G$ there is a scalar $\Delta(g) \in \mathbb{R}_{>0}$ such that $\int_G f(xy)\mathrm{d}\mu(x) = \Delta(y)\int_G f(x)\mathrm{d}\mu(x)$ for all $f \in \mathrm{L}^1(G)$. This equality does not depend on the choice of the Haar measure. The map $\Delta : G \to \mathbb{R}_{>0}$ is a continuous homomorphism.*

Proof. The positive linear functional $I : f \mapsto \int_G f(xg)\mathrm{d}\mu(x)$ is a Haar integral on $\mathrm{C_c}(G)$for every $g \in G$. Hence there is a scalar $\Delta(g) \in \mathbb{R}_{>0}$, which satisfies $I = \Delta(g)\int_G \mathrm{d}\mu$. Put differently, $\int_G f(xy)\mathrm{d}\mu(x) = \Delta(y)\int_G f(x)\mathrm{d}\mu(x)$ for all $f \in \mathrm{C_c}(G)$ and by approximation for all $f \in \mathrm{L}^1(G)$. Since the Haar measure is unique up to a scalar it follows that $\Delta(g)$ does not depend on its choice.

We show that $\Delta : G \to \mathbb{R}_{>0}$ is a group homomorphism. If $x, y \in G$ and $f \in \mathrm{C_c}(G)^+$ satisfies $\int f(z)\mathrm{d}\mu(z) = 1$, then

$$\Delta(xy) = \int_G f(zxy)\mathrm{d}\mu(z) = \Delta(y)\int_G f(zx)\mathrm{d}\mu(z) = \Delta(y)\Delta(x)$$

and

$$\Delta(x^{-1})\Delta(x) = \int_G f(zxx^{-1})\mathrm{d}\mu(z) = 1\,.$$

Let us next show that $\Delta : G \to \mathbb{R}_{>0}$ is continuous. According to Proposition 1.3.4 is suffices to check continuity at $e \in G$. Let $\varepsilon > 0$ and K a compact neighbourhood of $e \in G$. Let $V \subset K$ be a symmetric identity neighbourhood such that $|f(zx) - f(z)| < \frac{\varepsilon}{K \operatorname{supp} f}$ for all $x \in V$ and all $z \in G$. We used Lemma 4.1.2 saying that continuous compactly supported functions are uniformly continuous. We obtain that for $x \in V$

$$|\Delta(x) - 1| = \left|\int_G f(zx) - f(z)\mathrm{d}\mu(z)\right| < \frac{\varepsilon}{\mu(K \operatorname{supp} f)}\mu(V \operatorname{supp} f) \le \varepsilon\,.$$

This finishes the proof of the proposition. ☐

Definition 4.2.6. The homomorphism $\Delta : G \to \mathbb{R}_{>0}$ considered in Proposition 4.2.5 is called the Modular function of G. A locally compact group is called unimodular, if its modular function is trivial.

Let us now relate left and right Haar measure via the modular function. Before, we record one important approximation theorem.

Theorem 4.2.7. *Let X be a locally compact space with regular Radon measure μ. Then $\mathrm{C_c}(X) \subset \mathrm{L}^p(X,\mu)$ is dense for every $p \in [1,\infty)$.*

See Proposition 1.3.3 in [DE14] for a proof.

Proposition 4.2.8. *Let G be a locally compact group with a fixed Haar measure. For every $f \in \mathrm{L}^1(G)$, we have*

$$\int_G f(x)\mathrm{d}x = \int_G \Delta(x) f(x^{-1})\mathrm{d}x\,,$$

and for every $f \in \mathrm{C_c}(G)$ we have

$$\int_G f(x^{-1})\mathrm{d}x = \int_G \Delta(x) f(x)\mathrm{d}x\,.$$

Proof. Consider the positive functional $I(f) = \int_G \Delta(x)f(x^{-1})\mathrm{d}x$ defined on compactly supported continuous functions on G. It satisfies

$$\begin{aligned} I({}^z f) &= \int_G \Delta(x) f(z^{-1}x^{-1})\mathrm{d}x \\ &= \int_G \Delta(z)\Delta(xz^{-1}) f(x^{-1})\mathrm{d}x \\ &= \int_G \Delta(x) f(x^{-1})\mathrm{d}x \\ &= I(f)\,, \end{aligned}$$

for all $z \in G$. By uniqueness of the Haar integral (Theorem 4.2.4), it follows that I is a multiple of the Haar integral. Let $\varepsilon > 0$ and choose a symmetric neighbourhood U of e such that $|\Delta(x) - 1| < \varepsilon$ for all $x \in U$. Let $f \in \mathrm{C_c}(G)^+$ be a function supported in V satisfying $f(x^{-1}) = f(x)$ for all $x \in U$ and $\int_G f(x)\mathrm{d}x = 1$. Then

$$\begin{aligned} |I(f) - \int_G f(x)\mathrm{d}x| &= |\int_G f(x)\mathrm{d}x - \int_G \Delta(x)f(x^{-1})\mathrm{d}x| \\ &\leq \int_G |1-\Delta(x)| f(x)\mathrm{d}x \\ &\leq \varepsilon \|f\|_1 \\ &= \varepsilon\,. \end{aligned}$$

This shows that I equals the Haar integral. In particular, it extends to the $\mathrm{L}^1(G)$ and it hence defines the same functional on $\mathrm{L}^1(G)$. The second equality follows since for every $f \in \mathrm{C_c}(G)$ the function $x \mapsto f(x^{-1})$ is integrable. □

Exercise 4.2.9. Calculate the Haar measure and the modular function of $\mathbb{R}$ and of the $ax+b$-group $\mathbb{R} \rtimes \mathbb{R}_{>0}$. Observe that the modular function of $\mathbb{R} \rtimes \mathbb{R}_{>0}$ does not restrict to the modular function of $\mathbb{R}$.

Hint. The case $\mathbb{R}$ is classical, so let us consider the $ax+b$-group. We denote it by G. A short calculation checks that the product of Lebesgue measures on the underlying space $\mathbb{R} \times \mathbb{R}_{>0}$ of G is a right Haar measure. One deduces from Proposition 4.2.8 that he modular function of a locally compact group can be equally well calculated by means of the right Haar measure. Indeed if μ denotes a left Haar measure for G, then

$$\int_G f(yx^{-1})\mathrm{d}\mu(x) = \int_G \Delta(x)f(yx)\mathrm{d}\mu(x) = \int_G \Delta(y^{-1}x)f(x)\mathrm{d}\mu(x)$$
$$= \frac{1}{\Delta(y)}\int_G \Delta(x)f(x)\mathrm{d}\mu(x) = \frac{1}{\Delta(y)}\int_G f(x^{-1})\mathrm{d}\mu(x)\,.$$

Now a short calculation shows that $\Delta_{\mathrm{right}}(a,1) = 0$ and $\Delta_{\mathrm{right}}(0,b) = b$. Hence the modular function of G is given by $\Delta(a,b) = 1/b$. Using Proposition 4.2.8 again, we conclude that

$$\frac{1}{b}\mathrm{d}(\lambda_{\mathbb{R}} \times \lambda_{\mathbb{R}>0})(a,b)$$

is a left Haar measure for the $ax+b$-group. ☺

Exercise 4.2.10. Let G be a locally compact group admitting a compact open subgroup. Show that the modular function of G takes values in the positive rational numbers.

Hint. Let μ be a Haar measure of G and Δ its modular function. Fix a compact open subgroup $K \leq G$ and note that $\mu(K) < \infty$. Then $\Delta(g) = \frac{\mu(K)}{\mu(Kg)}$. Express the latter term by products and quotients of certain group indices. ☺

Exercise 4.2.11. Let $H \leq G$ be an open subgroup of a locally compact group. Then $\mu_G|_H$ is a Haar measure for H.

Hint. Note that $\mu_G|_H$ is non-zero and observe that left-invariance is inherited. ☺

4.3 The regular representation

In this section we are going to define and study the most important representation of a locally compact group. Let us start in the general setting of L^p-spaces.

Proposition 4.3.1. *Let G be a locally compact group and $p \in [1,\infty)$.*

(i) For all $g \in G$, the map $\mathrm{L}_g : \mathrm{C}_c(G) \to \mathrm{C}_c(G)$ defined by $(\mathrm{L}_g f)(x) = f(g^{-1}x)$ extends to an isometric operator on $\mathrm{L}^p(G)$.

(ii) For all $g \in G$, the map $\mathrm{R}_g : \mathrm{C}_c(G) \to \mathrm{C}_c(G)$ defined by $(\mathrm{R}_g f)(x) = f(xg)$ extends to the $\Delta(g)^{1/p}$-fold multiple of an isometric operator on $\mathrm{L}^p(G)$.

(iii) The maps $G \to \mathrm{GL}(\mathrm{L}^p(G)) : g \mapsto \mathrm{L}_g$ and $G \to \mathrm{GL}(\mathrm{L}^p(G)) : g \mapsto \mathrm{R}_g$ are group homomorphisms.

(iv) For all $f \in \mathrm{L}^p(G)$, the maps $g \mapsto \mathrm{L}_g f$ and $g \mapsto \mathrm{R}_g f$ are continuous from G into $\mathrm{L}^p(G)$.

(v) *The maps $G \to \mathrm{GL}(\mathrm{L}^p(G)) : g \mapsto \mathrm{L}_g$ and $G \to \mathrm{GL}(\mathrm{L}^p(G)) : g \mapsto \mathrm{R}_g$ are representations.*

Proof. Statement (i) follows from left-invariance of the Haar measure. By Theorem 4.2.7, it suffices to check that L_g is isometric on $\mathrm{C}_c(G)$. For all $g \in G$ and $f \in \mathrm{C}_c(G)$, we have

$$\|\mathrm{L}_g f\|_p^p = \int_G |f(g^{-1}x)|^p \mathrm{d}x = \int_G |f(x)|^p \mathrm{d}x = \|f\|_p^p .$$

We prove Statement (ii) in a similar way. For all $g \in G$ and $f \in \mathrm{C}_c(G)$, we have

$$\|\mathrm{R}_g f\|_p^p = \int_G |f(xg)|^p \mathrm{d}x = \Delta(g) \int_G |f(x)|^p \mathrm{d}x = \Delta(g)\|f\|_p^p .$$

Let us prove Statement statement (iii). An elementary calculation shows that $\mathrm{L}_{gh} f = \mathrm{L}_g \mathrm{L}_h f$ and $\mathrm{R}_{gh} f = \mathrm{R}_g \mathrm{R}_h f$ for all $g, h \in G$ and all $f \in \mathrm{C}_c(G)$ (iii). Further, it is clear that $\mathrm{L}_e = \mathrm{R}_e = \mathrm{id}_{\mathrm{L}^p(G)}$. By continuity the statement follows.

Let us show statement (iv). Since R and L are group homomorphisms, it suffices to check their continuity at the neutral element. We only treat the case of $x \mapsto \mathrm{L}_g f$, the other one being similar. First take $f \in \mathrm{C}_c(G)$. Let $\varepsilon > 0$. We show that there is a neighbourhood of the identity $U \subset G$ such that $\|\mathrm{L}_g f - f\|_p < \varepsilon$ for all $g \in U$. Fix a compact identity neighbourhood $K \subset G$. Since f is uniformly continuous by Lemma 4.1.2, there is a symmetric identity neighbourhood $U \subset K$ such $|f(gx) - f(x)|^p < \frac{\varepsilon}{\mu(K \operatorname{supp} f)}$ for all $g \in U$ and all $x \in G$. We obtain for $g \in U$:

$$\|\mathrm{L}_g f - f\|_p^p = \int_{U \operatorname{supp} f} |f(g^{-1}x) - f(x)|^p \mathrm{d}x \leq \mu(U \operatorname{supp} f) \frac{\varepsilon}{\mu(K \operatorname{supp} f)} \leq \varepsilon .$$

This shows that $g \mapsto \mathrm{L}_g f$ is continuous for $f \in \mathrm{C}_c(G)$.

Take an arbitrary $f \in \mathrm{L}^p(G)$ and let $\varepsilon > 0$. Find some $\tilde{f} \in \mathrm{C}_c(G)$ such that $\|f - \tilde{f}\|_p < \varepsilon/3$. There is some identity neighbourhood $U \subset G$ such that $\|\mathrm{L}_g \tilde{f} - \tilde{f}\| < \varepsilon/3$ for all $g \in U$. This implies for $g \in U$ □

$$\|\mathrm{L}_g f - f\| \leq \|\mathrm{L}_g f - \mathrm{L}_g \tilde{f}\|_p + \|\mathrm{L}_g \tilde{f} - \tilde{f}\|_p + \|\tilde{f} - f\|_p < \varepsilon .$$

From the previous proposition, we single out the particular case of $\mathrm{L}^2(G)$, which will be of main interest for us in this section.

Definition 4.3.2. Let G be a locally compact group. The unitary representations $\lambda, \rho : G \to \mathcal{U}(\mathrm{L}^2(G))$ defined by $\lambda_g = \mathrm{L}_g$ and $\rho_g = \Delta(g)^{-\frac{1}{2}} \mathrm{R}_g$ are called the left-regular representation and the right-regular representation of G.

Let us observe that we actually defined just one single representation up to unitary conjugacy.

Proposition 4.3.3. *Let G be a locally compact group. The left-regular representation and the right-regular representation of G are unitary equivalent.*

Proof. On compactly supported functions we define the operator $W : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ by $(Wf)(x) =$

$f(x^{-1})\Delta(x)^{1/2}$. Then Proposition 4.2.8 implies that

$$\begin{aligned}\|Wf\|_2^2 &= \int_G |f(x^{-1})\Delta(x)^{1/2}|^2 \mathrm{d}x \\ &= \int_G |f(x^{-1})|^2 \Delta(x) \mathrm{d}x \\ &= \int_G |f(x)|^2 \mathrm{d}x \\ &= \|f\|_2^2 .\end{aligned}$$

So W extends to a unitary operator $W : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$. For $g \in G$ and $f \in \mathrm{C}_\mathrm{c}(G)$ we have

$$\begin{aligned}W(\lambda_g f)(x) &= \lambda_g f(x^{-1})\Delta(x)^{1/2} \\ &= f(g^{-1}x^{-1})\Delta(x)^{1/2} \\ &= f((xg)^{-1})\Delta(xg)^{1/2}\Delta(g)^{-1/2} \\ &= (Wf)(xg)\Delta(g)^{-1/2} \\ &= \rho_g(Wf)(x) .\end{aligned}$$

This shows that $\lambda \cong \rho$. □

We now proceed to highlight the merits of the regular representation. It provides a natural mean to prove that every locally compact group has sufficiently many unitary representations.

Proposition 4.3.4. *The left-regular representation of a locally compact group is injective. In particular, every locally compact group has sufficiently many representations.*

Proof. Let G be a locally compact group and $g \in G$. Let U be an open neighbourhood of e such that U and gU are disjoint. If $f \in \mathrm{C}_\mathrm{c}(G)^+$ is any non-zero function whose support lies in U, then f and $\lambda_g f$ have disjoint support. It follows that

$$\|f - \lambda_g f\|_2^2 = \int_G |f(x) - f(g^{-1}x)|^2 \mathrm{d}x = \int_G |f(x)|^2 + |f(g^{-1}x)|^2 \mathrm{d}x = 2\|f\|_2^2 \neq 0 .$$

In particular, $\lambda_g \neq \mathrm{id}_{\mathrm{L}^2(G)}$. □

Proposition 4.3.5 (Fell's absorption principle). *Let G be a locally compact group. Denote by λ its left regular representation and let π be any unitary representation of G. Then $\lambda \otimes \pi$ is isomorphic to* $\dim \pi$ *copies of the left-regular representation: $\lambda \otimes \pi \cong \bigoplus_{\dim \pi} \lambda$.*

Proof. Let H be the Hilbert space on which G is represented by π. We make use of the identification $\mathrm{L}^2(G) \otimes H \cong \mathrm{L}^2(G, H)$. Consider the map $W : \mathrm{L}^2(G, H) \to \mathrm{L}^2(G, H)$ given by $(Wf)(g) = \pi(g)f(g)$ on functions $f \in \mathrm{C}_\mathrm{c}(G, H)$. This map gives rise to a well-defined isometry, since

$$\begin{aligned}\|Wf\|^2 &= \int_G \|Wf(g)\|^2 \mathrm{d}\mu(g) \\ &= \int_G \|\pi(g)f(g)\|^2 \mathrm{d}\mu(g) \\ &= \int_G \|f(g)\|^2 \mathrm{d}\mu(g) \\ &= \|f\|^2 ,\end{aligned}$$

for all $f \in \mathrm{C_c}(G, H)$. An inverse for W is given by $(W^{-1}f)(g) = \pi(g)^{-1}f(g)$, showing that W is a unitary. Further we have for $f \in \mathrm{C}(G, H)$ and $g, h \in G$

$$W(\lambda_g \otimes \mathrm{id})(f)(h) = \pi(h)(\lambda_g \otimes \mathrm{id})(f)(h) = \pi(h)f(g^{-1}h)$$

and

$$(\lambda_g \otimes \pi(g))Wf(h) = \pi(g)(Wf)(g^{-1}h) = \pi(g)\pi(g^{-1}h)f(g^{-1}h) = \pi(h)f(g^{-1}h)\,.$$

This shows that $W(\lambda_g \otimes \mathrm{id}) = (\lambda_g \otimes \pi(g))W$, which implies that

$$\lambda \otimes \pi \cong \lambda \otimes \mathrm{id}_H \cong \bigoplus_{\dim \pi} \lambda\,.$$

□

Remark 4.3.6. It would be interesting to find a characterisation of locally compact groups among all topological groups based on Fell's absorption property. Such a characterisation is not known to me. The following statement does *not* hold true: a topological group with sufficiently many unitary representations and a unitary representation satisfying Fell's absorption principle is already locally compact. A counterexample is the group $\mathrm{Sym}(\mathbb{N})$. Can we add natural conditions to the previous ones in order to obtain a characterisation of locally compact groups?

Let us end this section with the observation that the left- and the right-regular representation can be combined in an interesting way. It will mainly be of interest in Section 8 and more generally when studying so called type I groups.

Proposition 4.3.7. *Let G be a locally compact group. Then $(g, h) \mapsto \lambda_g \rho_h$ defines a unitary representation of $G \times G$ on $\mathrm{L}^2(G)$.*

Proof. For $g, h \in G$ and $f \in \mathrm{C_c}(G)$ we have

$$(\lambda_g \rho_h f)(x) = (\rho_h f)(g^{-1}x) = f(g^{-1}xh)\Delta(h)^{-1/2} = (\rho_h \lambda_g f)(x)\,.$$

Hence the two unitary representations λ and ρ on $\mathrm{L}^2(G)$ commute. The universal property of the product construction now provides us with a well-defined continuous homomorphism $\lambda \times \rho : G \times G \to \mathcal{U}(\mathrm{L}^2(G))$. □

5 Bochner integrals

We only give the statement of existence and uniqueness of a Banach space valued integral, as we will need it in the sequel. The reference [DE14, B.6 p.302ff] gives more details and an excellent exposition on the matter.

Theorem 5.0.1. *Let V be a Banach space and (X, μ) a measure space. If $f : X \to V$ is a measurable function such that $x \mapsto \|f(x)\|$ is an integrable function, then there is a unique element $\int_X f \mathrm{d}\mu \in V$ satisfying*

$$\varphi(\int_X f \mathrm{d}\mu) = \int_X \varphi \circ f \mathrm{d}\mu$$

for all functionals $\varphi \in V^*$. Further, we have

$$\| \int_X f(x) \mathrm{d}\mu(x) \| \leq \int_X \|f(x)\| \mathrm{d}\mu(x) \, .$$

We are mostly going to use Bochner integrals in the context of unitary representations. The following lemma allows us to reduce some technicalities in Section 6.

Lemma 5.0.2. *Let G be a locally compact group, (π, H) a unitary representation of G and $f \in \mathrm{L}^1(G)$. Define $T_f = \int_G f(x)\pi(x)\mathrm{d}x \in \mathcal{B}(H)$. For all $\xi \in H$ we have*

$$T_f \xi = \int_G f(x)\pi(x)\xi \mathrm{d}x \in H \, .$$

Proof. Since $\|\pi(x)\| = 1$ and $\|\pi(x)\xi\| = \|\xi\|$ for all $x \in G$ and all $\xi \in H$, the functions $x \mapsto \|\pi(x)\|$ and $x \mapsto \|f(x)\pi(x)\xi\|$ are integrable. So all expressions in the statement of the lemma are well-defined.

Let $\xi \in H$. For all $\eta \in H$, the map $T \mapsto \langle T\xi, \eta \rangle$ is a linear functional on $\mathcal{B}(H)$. Theorem 5.0.1 hence implies that

$$\begin{aligned} \langle T_f \xi, \eta \rangle &= \int_G f(x) \langle \pi(x)\xi, \eta \rangle \mathrm{d}x \\ &= \int_G \langle f(x)\pi(x)\xi, \eta \rangle \mathrm{d}x \\ &= \langle \int_G f(x)\pi(x)\xi \mathrm{d}x, \eta \rangle \, . \end{aligned}$$

Since $\eta \in H$ is arbitrary, this show that $T_f \xi = \int_G f(x)\pi(x)\xi \mathrm{d}x$ as elements of H. ☐

6 Group algebras

The existence of a Haar measure on a locally compact group allows us to define a number of algebras. Their definition is always based on the following convolution product.

Definition 6.0.1. Let $f, g : G \to \mathbb{C}$ be measurable functions on a locally compact group G.

- The convolution $f * g$ is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\mathrm{d}y\,,$$

  whenever this integral makes sense.

- There is an involution $f \mapsto f^*$ defined by $f^*(x) = \Delta(x)\overline{f(x^{-1})}$.

We can now define the first algebra associated with a locally compact group G.

Theorem 6.0.2. *Let G be a locally compact group. Then the vector space $\mathrm{C}_c(G)$ of continuous compactly supported functions on G becomes a *-algebra with the convolution product and involution as described in Definition 6.0.1*

Proof. For $f, g \in \mathrm{C}_c(G)$, the function $f * g$ is well-defined, since $(x, y) \mapsto f(y)g(y^{-1}x)$ is a compactly supported continuous function. Further $f * g \in \mathrm{C}_c(G)$, since the compactly supported continuous function $(x, y) \mapsto f(y)g(y^{-1}x)$ on the locally compact group $G \times G$ is uniformly constant by Lemma 4.1.2. There remain two non-trivial assertions of the theorem. First, the convolution product is associative. Second, we have $(f * g)^* = g^* * f^*$ for all $f, g \in \mathrm{C}_c(G)$. We leave associativity as an exercise and show that $(f * g)^* = g^* * f^*$ for all $f, g \in \mathrm{C}_c(G)$. This is done by the following two calculations.

$$(f * g)^*(x) = \overline{(f * g)(x^{-1})}\Delta(x) = \Delta(x) \int_G \overline{f(y)g(y^{-1}x^{-1})}\mathrm{d}y$$

$$\begin{aligned} g^* * f^*(x) &= \int_G g^*(y)f^*(y^{-1}x)\mathrm{d}y \\ &= \int_G \overline{g(y^{-1})}\Delta(y)\overline{f(x^{-1}y)}\Delta(y^{-1}x)\mathrm{d}y \\ &= \Delta(x) \int_G \overline{g(y^{-1})f(x^{-1}y)}\mathrm{d}y \\ &= \Delta(x) \int_G \overline{g(y^{-1}x^{-1})f(y)}\mathrm{d}y\,. \end{aligned}$$

This shows the desired equality and finishes the proof of the theorem. □

6.1 The Banach *-algebra $\mathrm{L}^1(G)$

Recognising that $\mathrm{C}_c(G)$ naturally forms a *-algebra for a locally compact group is an important step to make use of functional analytic techniques. However, infinite dimensional algebras are very hard to understand if they are not equipped with a suitable topology. We are hence going to present completions of $\mathrm{C}_c(G)$ in various norms. We start with the familiar $\|\ \|_1$-norm.

Definition 6.1.1. A Banach *-algebra A is a *-algebra with a Banach norm such that $\|ab\| \le \|a\|\|b\|$ and $\|a^*\| = \|a\|$ for all $a, b \in A$.

Theorem 6.1.2. *Let G be a locally compact group.*

(i) Let $f, g \in \mathrm{L}^1(G)$. Then $f * g$ is well-defined almost everywhere and it satisfies $\|f * g\|_1 \le \|f\|_1\|g\|_1$.

(ii) For $f \in \mathrm{L}^1(G)$, we have $\|f^*\|_1 = \|f\|_1$.

*(iii) The convolution and the involution turn $\mathrm{L}^1(G)$ into a Banach *-algebra.*

Proof. Let $f, g \in \mathrm{C_c}(G)$. The function $G \times G \ni (x, y) \mapsto f(y)g(y^{-1}x)$ satisfies

$$\int_G \int_G |f(y)g(y^{-1}x)|\mathrm{d}x\mathrm{d}y = \int_G |f(y)g(x)|\mathrm{d}x\mathrm{d}y = \|f\|_1\|g\|_1 < \infty .$$

The Fubini theorem hence tells us that the integral $\int_G f(y)g(y^{-1}x)\mathrm{d}y$ is well-defined for almost every $x \in G$. So $f * g$ is well-defined almost everywhere. Further,

$$\begin{aligned}
\|f * g\|_1 &= \int_G |\int_G f(y)g(y^{-1}x)\mathrm{d}y|\mathrm{d}x \\
&\le \int_G \int_G |f(y)g(y^{-1}x)|\mathrm{d}y\mathrm{d}x \\
&= \int_G \int_G |f(y)g(y^{-1}x)|\mathrm{d}x\mathrm{d}y \\
&= \int_G \int_G |f(y)||g(x)|\mathrm{d}x\mathrm{d}y \\
&= \|f\|_1\|g\|_1 .
\end{aligned}$$

This shows that the convolution product of $\mathrm{C_c}(G)$ extends uniquely to a algebra product on $\mathrm{L}^1(G)$. So the the first statement of the theorem is proven.

For $f \in \mathrm{L}^1(G)$, we have thanks to Proposition 4.2.8

$$\|f^*\|_1 = \int_G |\overline{f(x^{-1})}\Delta(x)|\mathrm{d}x = \int_G |f(x)|\mathrm{d}x = \|f\|_1 .$$

This shows the second statement of the theorem.

Since $\mathrm{C_c}(G)$ is an algebra, (i) shows that $\mathrm{L}^1(G)$ is a Banach algebra with the convolution product. Further, $\mathrm{C_c}(G)$ is a *-algebra, whose involution extends uniquely to $\mathrm{L}^1(G)$ by (ii). This shows that $\mathrm{L}^1(G)$ is a Banach *-algebra. □

Remark 6.1.3. Our definition $f^*(g) = \Delta(g)\overline{f(g^{-1})}$ together with the requirement $\|f^*\|_1 = \|f\|_1$ forces Δ to be defined by $\int_G f(xy)\mathrm{d}(x) = \Delta(y)\int_G f\mathrm{d}(x)$, as we have done earlier.

For later use, we fix the following alternative way to write the convolution of two L^1-functions

Lemma 6.1.4. *Let G be a locally compact group and $f, g \in \mathrm{L}^1(G)$. Then $f * g = \int_G f(x)\mathrm{L}_x g\mathrm{d}x$.*

Proof. Identify the dual space of $\mathrm{L}^1(G)$ with the essentially bounded functions $\mathrm{L}^\infty(G)$. Theorem 5.0.1 implies that for $h \in \mathrm{L}^\infty(G)$ and $f, g \in \mathrm{L}^1(G)$ we have

$$\begin{aligned}\langle \int_G f(x)\mathrm{L}_x g \mathrm{d}x, h\rangle &= \int_G \langle f(x)\mathrm{L}_x g, h\rangle \mathrm{d}x \\ &= \int_G \int_G f(x)\mathrm{L}_x g(y)\overline{h(y)}\mathrm{d}y\mathrm{d}x \\ &= \int_G \int_G f(x)g(x^{-1}y)\overline{h(y)}\mathrm{d}y\mathrm{d}x \\ &= \int_G \int_G f(x)g(x^{-1}y)\mathrm{d}x\overline{h(y)}\mathrm{d}y \\ &= \int_G (f * g)(y)\overline{h(y)}\mathrm{d}y \\ &= \langle f * g, h\rangle \,.\end{aligned}$$

This shows $\int_G f(x)\mathrm{L}_x g\mathrm{d}x = f * g$. □

6.2 Approximate identities and Dirac nets

In [DE14] the notion of Dirac nets of continuous functions is used to establish essential links between a locally compact group G and its L^1-algebra. We use a slightly more general concept here.

Definition 6.2.1. Let G be a locally compact group with fixed Haar measure μ. A *Dirac function* on G is a non-negative element $f \in \mathrm{L}^1(G)$ such that $\int f \mathrm{d}\mu = 1$. Given a neighbourhood basis $\mathcal{U}$ of the identity $e \in G$, a Dirac net for $\mathcal{U}$ is a family of Dirac functions f_U, $U \in \mathcal{U}$ such that $\operatorname{supp} f \subset U$, where $\operatorname{supp} f = \bigcap\{V^{\mathrm{c}} \mid V \subset G \text{ open and } f|_V = 0 \in \mathrm{L}^1(G)\}$ denotes the essential support of f.

Definition 6.2.2. Let G be a locally compact group and $f \in \mathrm{L}^1(G)$ a Dirac function. Let (π, H) be a unitary representation of G. Then the averaging operator associated with f in π is defined as the Bochner integral

$$T_f = \int_G f(x)\pi(x)\mathrm{d}x \,.$$

If $U \subset G$ a measurable subset of non-zero finite Haar measure $0 \neq \mu(U) < \infty$ and $f = \frac{1}{\mu(U)}\mathbb{1}_U$, then we also write $T_f = T_U$.

Definition 6.2.3. Let A be a normed algebra. An approximate identity for A is a net of elements $(a_i)_{i\in I}$ such that $a_i b \to b$ and $b a_i \to b$ for all $b \in A$.

Lemma 6.2.4. *Let $(f_U)_{U\in\mathcal{U}}$ be a Dirac net for a locally compact group G.*

- *$(f_U)_{U\in\mathcal{U}}$ is an approximate identity for $\mathrm{L}^1(G)$.*
- *If (π, H) is a unitary representation of G and $T_{f_U}, U \in \mathcal{U}$ are the associated averaging operators, we have $T_{f_U} \to \mathrm{id}$ in the strong operator topology of $\mathcal{B}(H)$ as $U \downarrow \{e\}$.*

Proof. We start by showing that $(f_U)_{U\in\mathcal{U}}$ is an approximate identity for $\mathrm{L}^1(G)$. Let $g \in \mathrm{C_c}(G)$ and $\varepsilon > 0$. Fix some compact identity neighbourhood $K \subset G$. Because g is uniformly continuous by Lemma 4.1.2, there is an identity neighbourhood $V \subset K$ such that $|g(x) - g(y^{-1}x)| < \varepsilon/\mu(K \operatorname{supp} g)$ for all $y \in V$ and all $x \in G$. For every $U \in \mathcal{U}$ satisfying $U \subset V$, we obtain the following estimate.

$$\begin{aligned}
\|f_U * g - g\|_1 &= \int_G \left| \Big(\int_G f_U(y) g(y^{-1}x) \mathrm{d}y \Big) - g(x) \right| \mathrm{d}x \\
&= \int_{K \operatorname{supp} g} \left| \int_G f_U(y)(g(y^{-1}x) - g(x)) \mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_{K \operatorname{supp} g} \int_G f_U(y) |g(y^{-1}x) - g(x)| \mathrm{d}y \mathrm{d}x \\
&\leq \mu(K \operatorname{supp} g) \frac{\varepsilon}{\mu(K \operatorname{supp} g)} \\
&= \varepsilon .
\end{aligned}$$

This shows that $f_U * g \to g$ for all $g \in \mathrm{C_c}(G)$ and hence for all $g \in \mathrm{L}^1(G)$, because $\|f_U\|_1 = 1$ for all $U \in \mathcal{U}$. Further, $g * f_U = (f_U^* * g^*)^* \to (g^*)^* = g$, since $(f_U^*)_{U\in\mathcal{U}}$ is a Dirac net for the neighbourhood basis $\{U^{-1} \mid U \in \mathcal{U}\}$ of $e \in G$. This proves the first statement of the lemma.

We show the second statement of the lemma. Let $\xi_1, \dots, \xi_n \in H$ and let $\varepsilon > 0$. Since $\pi : G \to \mathcal{U}(H)$ is strongly continuous, there is an identity neighbourhood $V \subset G$ such that $\|\pi(x)\xi_i - \xi_i\| < \varepsilon$ for all $x \in V$ and all $i \in \{1, \dots, n\}$. For $\eta \in H$ and $i \in \{1, \dots, n\}$ we calculate

$$\begin{aligned}
|\langle T_{f_U}\xi_i - \xi_i, \eta\rangle| &= |\int_G f(x)\langle \pi(x)\xi_i, \eta\rangle \mathrm{d}x - \langle \xi_i, \eta\rangle| \\
&= |\int_G f(x)\langle \pi(x)\xi_i - \xi_i, \eta\rangle \mathrm{d}x| \\
&\leq \int_G f(x)|\langle \pi(x)\xi_i - \xi_i, \eta\rangle| \mathrm{d}x \\
&\leq \varepsilon\|\eta\| .
\end{aligned}$$

This shows that $\|T_{f_U}\xi_i - \xi_i\| \leq \varepsilon$ for all $i \in \{1, \dots, n\}$, which finishes the proof of the lemma. □

6.3 Integration and desintegration of representations

In this section we are going to set up a 1-1 correspondence between unitary representations of a locally compact group G and a certain class of contractive *-representations of $\mathrm{L}^1(G)$. The correct class of *-representations to be considered is indicated by Lemma 6.2.4, which showed a certain non-degeneracy of the action of Dirac nets via unitary representations.

Definition 6.3.1. Let A be an algebra and a $\pi : A \to \mathcal{B}(V)$ be a representation on a Banach space. Then π is called non-degenerate, if the linear span of $\pi(A)V$ is dense in V.

Relating the previous definition to Lemma 6.2.4, we note the following theorem in passing.

Theorem 6.3.2. *Let A be a normed algebra with an approximate identity and let $\pi : A \to \mathcal{B}(V)$ be a representation of A on a Banach space V. Then the following statements are equivalent.*

- *π is non-degenerate.*
- *$\pi(a_i) \to \mathrm{id}_V$ strongly for all bounded approximate identities $(a_i)_i$ of A.*
- *$\pi(a_i) \to \mathrm{id}_V$ strongly for some bounded approximate identity $(a_i)_i$ of A.*

Proposition 6.3.3. *Let (π, H) be a unitary representation of a locally compact group G. Then $f \mapsto \int_G f(x)\pi(x)\mathrm{d}x$ defines a non-degenerate contractive *-homomorphism $\mathrm{L}^1(G) \to \mathcal{B}(H)$.*

Proof. Since $\|\pi(x)\| = 1$ for every $x \in G$, the function $x \mapsto f(x)\pi(x)$ is integrable for every $f \in \mathrm{L}^1(G)$. Further, $\|\int_G f(x)\pi(x)\mathrm{d}x\| \leq \|f\|_1$ follows from the general properties of the Bochner integral. So $f \mapsto \int_G f(x)\pi(x)\mathrm{d}x$ is well-defined contractive linear map from $\mathrm{L}^1(G)$ to $\mathcal{B}(H)$, which we will denote by π.

For all $f, g \in \mathrm{L}^1(G)$ and $\xi, \eta \in H$, we have

$$\begin{aligned}
\langle \pi(f * g)\xi, \eta \rangle &= \int_G \langle (f * g)(x)\pi(x)\xi, \eta \rangle \mathrm{d}x \\
&= \int_G (f * g)(x)\langle \pi(x)\xi, \eta \rangle \mathrm{d}x \\
&= \int_G \int_G f(y)g(y^{-1}x)\mathrm{d}y \langle \pi(x)\xi, \eta \rangle \mathrm{d}x \\
&= \int_G \int_G f(y)g(y^{-1}x)\langle \pi(x)\xi, \eta \rangle \mathrm{d}x\mathrm{d}y \\
&= \int_G \int_G f(y)g(x)\langle \pi(y)\pi(x)\xi, \eta \rangle \mathrm{d}x\mathrm{d}y \\
&= \int_G \int_G f(y)g(x)\langle \pi(x)\xi, \pi(y)^*\eta \rangle \mathrm{d}x\mathrm{d}y \\
&= \int_G f(y)\langle \pi(g)\xi, \pi(y)^*\eta \rangle \mathrm{d}y \\
&= \int_G f(y)\langle \pi(y)\pi(g)\xi, \eta \rangle \mathrm{d}y \\
&= \langle \pi(f)\pi(g)\xi, \eta \rangle .
\end{aligned}$$

This shows that $\pi(f * g) = \pi(f)\pi(g)$. So π is a representation. Next, for $f \in \mathrm{L}^1(G)$ and for $\xi, \eta \in H$

we have

$$
\begin{aligned}
\langle \pi(f)\xi, \eta\rangle &= \int_G f(x)\langle \pi(x)\xi, \eta\rangle \mathrm{d}x \\
&= \int_G f(x)\langle \xi, \pi(x)^*\eta\rangle \mathrm{d}x \\
&= \int_G f(x)\overline{\langle \pi(x^{-1})\eta, \xi\rangle} \mathrm{d}x \\
&= \overline{\int_G \Delta(x)\overline{f(x^{-1})}\langle \pi(x)\eta, \xi\rangle \mathrm{d}x} \\
&= \overline{\int_g f^*(x)\langle \pi(x)\eta, \xi\rangle \mathrm{d}x} \\
&= \overline{\langle \pi(f^*)\eta, \xi\rangle} \\
&= \langle \xi, \pi(f^*)\eta\rangle \, .
\end{aligned}
$$

This shows that π is a $*$-representation.

If $(f_U)_{U\in\mathcal{U}}$ is a Dirac net in G, the Lemma 6.2.4 says that $\pi(f_U) \to \mathrm{id}_H$ strongly. It follows that $\operatorname{span} \pi(\mathrm{L}^1(G))H \subset H$ is dense, meaning that $\pi : \mathrm{L}^1(G) \to \mathcal{B}(H)$ is non-degenerate. □

Definition 6.3.4. If (π, H) is a unitary representation of a locally compact group G, then the $*$-representation $\pi : \mathrm{L}^1(G) \to \mathcal{B}(H)$ constructed in Proposition 6.3.3 is called the integrated representation of (π, H).

The next proposition establishes the fact that there is a correspondence between unitary representations of a locally compact group G and non-degenerate continuous $*$-representations of $\mathrm{L}^1(G)$.

Proposition 6.3.5. *Let G be a locally compact group and let $\pi : \mathrm{L}^1(G) \to \mathcal{B}(H)$ be a non-degenerate continuous $*$-representation on the bounded operators on a Hilbert space. Then there is unique unitary representation of G, whose integrated representation is π.*

Proof. By assumption the linear space spanned by $\pi(\mathrm{L}^1(G))H$ is dense in H. For $x \in G$, we define an operator $\pi(x)$ on this space by

$$\pi(x)\sum_i \pi(f_i)\xi_i := \sum_i \pi(L_x f_i)\xi_i \, .$$

We show that $\pi(g)$ is well-defined and extends to a unitary operator on H. To this end note that for

$f, g \in \mathrm{C_c}(G)$ and $y \in G$, we have

$$\begin{aligned}
(g^* * \mathrm{L}_x f)(y) &= \int_G g^*(z)(\mathrm{L}_x f)(z^{-1}y)\mathrm{d}z \\
&= \int_G \overline{g(z^{-1})}\Delta(z) f(x^{-1}z^{-1}y)\mathrm{d}z \\
&= \int_G \overline{g((zx^{-1})^{-1})}\Delta(zx^{-1}) f(x^{-1}(zx^{-1})^{-1}y)\Delta(x)\mathrm{d}z \\
&= \int_G \overline{g(xz^{-1})}\Delta(z) f(z^{-1}y)\mathrm{d}z \\
&= \int_G \overline{(\mathrm{L}_{x^{-1}} g)(z^{-1})}\Delta(z) f(z^{-1}y)\mathrm{d}z \\
&= ((\mathrm{L}_{x^{-1}} g)^* * f)(y) .
\end{aligned}$$

It follows that that $g^* * \mathrm{L}_x f = (\mathrm{L}_{x^{-1}} g)^* * f$ for all $f, g \in \mathrm{L}^1(G)$. Now take $\sum_i \pi(f_i)\xi_i \in \pi(\mathrm{L}^1(G)H)$. We obtain

$$\begin{aligned}
\| \sum_i \pi(L_x f_i)\xi_i \|^2 &= \sum_{i,j} \langle \pi(\mathrm{L}_x f_i)\xi_i, \pi(\mathrm{L}_x f_j)\xi_j \rangle \\
&= \sum_{i,j} \langle \pi(\mathrm{L}_x f_j)^* \pi(\mathrm{L}_x f_i)\xi_i, \xi_j \rangle \\
&= \sum_{i,j} \langle \pi((\mathrm{L}_x f_j)^* * \mathrm{L}_x f_i)\xi_i, \xi_j \rangle \\
&= \sum_{i,j} \langle \pi((\mathrm{L}_{x^{-1}} \mathrm{L}_x f_j)^* * f_i)\xi_i, \xi_j \rangle \\
&= \sum_{i,j} \langle \pi(f_j^* * f_i)\xi_i, \xi_j \rangle \\
&= \sum_{i,j} \langle \pi(f_i)\xi_i, \pi(f_j)\xi_j \rangle \\
&= \| \sum_i \pi(f_i)\xi_i \|^2 .
\end{aligned}$$

So $\pi(x)$ is well-defined and extends to a unitary operator on H. We next show that $x \mapsto \pi(x) \in \mathcal{U}(H)$ is continuous. It suffices to verify pointwise continuity on the linear span of $\pi(\mathrm{L}^1(G))H$. Let $f_1, \ldots, f_n \in \mathrm{L}^1(G)$ and let $\xi_1, \ldots, \xi_n \in H$ be unit vectors. Let $\varepsilon > 0$. By Proposition 4.3.1 (iv) there is an identity neighbourhood $U \subset G$ such that $\|\mathrm{L}_x f_i - f_i\|_1 < \frac{\varepsilon}{n}$ for all $i \in \{1, \ldots, n\}$ and all $x \in U$. Hence we obtain for $x \in U$

$$\begin{aligned}
\|\pi(x)\sum_i \pi(f_i)\xi_i - \sum_i \pi(f_i)\xi_i\|_1 &\leq \sum_i \|\pi(\mathrm{L}_x f_i - f_i)\xi_i\|_1 \\
&\leq \sum_i \|\mathrm{L}_x f_i - f_i\|_1 \\
&\leq n\frac{\varepsilon}{n} \\
&= \varepsilon .
\end{aligned}$$

This shows continuity of $\pi : G \to \mathcal{U}(H)$, so that it is indeed a unitary representation.

Denote by $\tilde{\pi}$ the integrated representation of $\pi : G \to \mathcal{U}(H)$. We show that $\tilde{\pi} = \pi$ as representations of $\mathrm{L}^1(G)$. It suffices to show that $\tilde{\pi}(f)\pi(g)\xi = \pi(f)\pi(g)\xi$ for all $f, g \in \mathrm{L}^1(G)$ and all $\xi \in H$. This follows from the next calculation using Lemmas 5.0.2 and 6.1.4.

$$\begin{aligned}\tilde{\pi}(f)\pi(g)\xi &= \int_G f(x)\pi(x)\pi(g)\xi \mathrm{d}x \\ &= \int_G f(x)\pi(\mathrm{L}_x g)\xi \mathrm{d} \\ &= \pi(\int_G f(x)\mathrm{L}_x g \mathrm{d}x)\xi \\ &= \pi(f * g)\xi\,.\end{aligned}$$

□

We summarise the content of this section in the following theorem.

Theorem 6.3.6. *Let G be a locally compact group. There is a one-to-one correspondence between unitary representations of G and non-degenerate *-representations of $\mathrm{L}^1(G)$, which assigns to a unitary representation (π, H) its integrated form $f \mapsto \int_G f(x)\pi(x)\mathrm{d}x$.*

6.4 Group C*-algebras

We are next going to consider C*-completions of $\mathrm{L}^1(G)$. The reason is that C*-algebras have a much better general structure theory than general Banach-*-algebras. Let us recall the definition of a C*-algebra and one important characterisation.

Definition 6.4.1. A C*-algebra is a Banach *-algebra A that satisfies the C*-identity $\|x^*x\| = \|x\|^2$ for all $x \in A$.

Theorem 6.4.2. *Let A be Banach *-algebra. Then A is a C*-algebra if and only if it admits an isometric *-representation into $\mathcal{B}(H)$ for some Hilbert space H.*

The first C*-completion that we introduce naturally arises from the regular representation of G. Its definition is justified by the central role of the regular representation as well as the subsequent proposition showing that we obtain an actually obtain a completion of $\mathrm{L}^1(G)$.

Definition 6.4.3. Let G be a locally compact group and $\lambda : \mathrm{L}^1(G) \to \mathcal{B}(\mathrm{L}^2(G))$ the integrated form of the left-regular representation. Then $\mathrm{C}^*_{\mathrm{red}}(G) = \overline{\lambda(\mathrm{L}^1(G))}^{\|\cdot\|}$ is the reduced group C*-algebra of G.

Proposition 6.4.4. *The *-homomorphism $\lambda : \mathrm{L}^1(G) \to \mathrm{C}^*_{\mathrm{red}}(G)$ is injective.*

Proof. First observe that by the definition of convolution and integrated representations, we have the identity $\lambda(f)g = f * g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ for all $f \in \mathrm{L}^1(G)$ and $g \in \mathrm{L}^2(G) \cap \mathrm{L}^1(G)$. Take $f \in \mathrm{L}^1(G)$ such that $\lambda(f) = 0$. If $(f_U)_{U \in \mathcal{U}}$ is a Dirac net in $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, then

$$0 = \lambda(f)f_U = f * f_U \xrightarrow{\|\ \|_1} f\,,$$

By Lemma 6.2.4. This shows that $f = 0$. □

The second C*-completion of $\mathrm{L}^1(G)$ that we introduce is the maximal group C*-algebra. It captures the unitary representation theory of G in the same fashion as $\mathrm{L}^1(G)$ does according to Theorem 6.3.6.

Definition 6.4.5. Let G be a locally compact group. For $f \in L^1(G)$ define the universal C*-norm by $\|f\|_{C^*_{max}} = \sup\{\|\pi(f)\| \mid \pi \text{ unitary representation of } G\}$. The completion of $L^1(G)$ with respect to this norm is called the universal C*-algebra of G and it is denoted by $C^*_{max}(G)$.

Theorem 6.4.6. *Let G be a locally compact group. There is a one-to-one correspondence between unitary representations of G and non-degenerate *-representations of $C^*_{max}(G)$.*

Proof. In view of Theorem 6.3.6 it suffices to argue that there is a one-to-one correspondence between representations of $L^1(G)$ and $C^*_{max}(G)$. This follows from the fact that for every non-degenerate *-representation $\pi : L^1(G) \to \mathcal{B}(H)$, we have $\|\pi(f)\| \leq \|f\|_{C^*_{max}}$, so that π uniquely extends to a non-degenerate *-representation of $C^*_{max}(G)$. Vice versa, every non-degenerate *-representation of $C^*_{max}(G)$ restricts to a non-degenerate *-representation of its dense subalgebra $L^1(G)$. □

Remark 6.4.7. Since the C*-norm on $L^1(G) \subset C^*_{max}(G)$ dominates the norm of $L^1(G) \subset C^*_{red}(G)$, the inclusion $\lambda : L^1(G) \hookrightarrow C^*_{red}(G)$ extends to a natural surjective *-homomorphism $\lambda : C^*_{max}(G) \to C^*_{red}(G)$.

6.5 Group von Neumann algebras

In this section we complete the reduced group C*-algebra to the group von Neumann algebra. The advantage of this step is a gain of flexibility in the bigger von Neumann algebra.

Definition 6.5.1. A *von Neumann algebra* is a unital strongly closed *-subalgebra $M \subset \mathcal{B}(H)$ for some Hilbert space H.

Theorem 6.5.2. *Let H be a Hilbert space and $A \subset \mathcal{B}(H)$ a unital *-subalgebra. Then the following statements are equivalent.*

- *$A = A''$.*
- *A is strongly closed.*
- *A is weakly closed.*

Remark 6.5.3.

- There is a completely intrinsic characterisation of a von Neumann algebra, not referring to a concrete Hilbert space representation. A C*-algebra M is a von Neumann algebra if and only if it admits an isometric predual as a Banach space. This is known as Sakai's theorem.
- The theory of von Neumann algebras is sometimes considered *noncommutative measure theory*. Indeed, abelian von Neumann algebras are of the form $L^\infty(X)$ for a standard measure space X. This perspective gives a foundation to the claim that von Neumann algebras are more flexible than C*-algebras, which are of topological nature. This can be illustrated by the fact that if $x \in M$ is an operator in a von Neumann algebra and $x = v|x|$ is its polar decomposition, then $v, |x| \in M$. In a C*-algebraic setting, the partial isometry v does not necessarily remain in the algebra.

The following definition and the subsequent theorem demonstrate the flexibility of the von Neumann algebraic setting.

Definition 6.5.4. Let G be a locally compact group. The *group von Neumann algebra* of G is defined as $L(G) = \pi(G)''$.

Theorem 6.5.5. *Let G be a locally compact group. Then $L(G) = \overline{C^*_{red}(G)}^{SOT}$.*

6.6 Some properties of group algebras

In this section, we briefly mention two properties of a group that can be characterised by their group algebras.

Theorem 6.6.1. *Let G be a locally compact group. Then the following statements are equivalent.*

(i) G *is discrete.*

(ii) $\mathrm{C_c}(G)$ *is unital.*

(iii) $\mathrm{L}^1(G)$ *is unital.*

(iv) $\mathrm{C}^*_{\mathrm{red}}(G)$ *is unital.*

(v) $\mathrm{C}^*_{\max}(G)$ *is unital.*

Remark 6.6.2. There is no characterisation of discrete groups in terms of their group von Neumann algebras possible. In fact, we will see that $\mathrm{L}(\mathbb{Z}) \cong \mathrm{L}^\infty(\mathrm{S}^1)$ and $\mathrm{L}(\mathbb{R}) \cong \mathrm{L}^\infty(\mathbb{R})$ in Section 9.6.3. These von Neumann algebras are isomorphic, since $\mathrm{S}^1 \setminus \{1\}$ is homeomorphic with $\mathbb{R}$ and $\{1\} \subset \mathrm{S}^1$ is a subset of Lebesgue measure 0.

Theorem 6.6.3. *Let G be a locally compact group. Then the following statements are equivalent.*

(i) G *is abelian.*

(ii) $\mathrm{C_c}(G)$ *is abelian.*

(iii) $\mathrm{L}^1(G)$ *is abelian.*

(iv) $\mathrm{C}^*_{\mathrm{red}}(G)$ *is abelian.*

(v) $\mathrm{C}^*_{\max}(G)$ *is abelian.*

(vi) $\mathrm{L}(G)$ *is abelian.*

7 Compact operators, Hilbert-Schmidt operators and trace class operators

In this section we develop the necessary theory of compact operators and Hilbert-Schmidt operators as they will be used in the representation theory of compact groups.

7.1 Compact operators and the spectral theorem

Definition 7.1.1. Let H be a Hilbert space. An operator $T \in \mathcal{B}(H)$ is called compact, if $TB \subset H$ is relatively compact for every bounded set $B \subset H$. The set of all compact operators in $\mathcal{B}(H)$ is denoted by $\mathcal{K}(H)$.

The spectral theorem for compact operators draws a strong parallel between compact operators and operators on finite dimensional Hilbert spaces. It is the content of a functional analysis course. We cite the formulation stated in [Con90, Theorem 7.9]. Recall that an operator $T \in \mathcal{B}(H)$ is called normal if $T^*T = TT^*$.

Theorem 7.1.2. *Every compact normal operator is diagonalisable. That is, if $T \in \mathcal{K}(H)$ is a compact normal operator on a Hilbert space, then the spectrum of $\sigma(T)$ is countable, every element of $\sigma(T)$ is an eigenvalue of T and $H = \ker T \oplus \bigoplus_{\lambda \in \sigma(T) \setminus \{0\}} \mathrm{Eig}(T, \lambda)$, where $\mathrm{Eig}(T, \lambda) = \{\xi \in H \mid T\xi = \lambda\xi\}$ is the finite dimensional eigenspace. The so called spectral projections E_λ, $\lambda \in \sigma(T)$ onto the subspace $\mathrm{Eig}(T, \lambda) \leq H$ lie in the C^*-algebra generated by T. The projection E_0 onto the kernel of T lies in the von Neumann algebra generated by T.*

We will need the following characterisation of compact operators later on.

Proposition 7.1.3. *Let H be a Hilbert space. For $T \in \mathcal{B}(H)$ the following conditions are equivalent.*

- *T is compact*
- *For every orthonormal basis $(e_i)_{i \in I}$ of H we have $\|Te_i\| \to 0$.*
- *There are finite rank projections $(P_n)_{n \in \mathbb{N}}$ such that $\|T - TP_n\| \to 0$.*

Proof. Assume that T is compact and let $(e_i)_{i \in I}$ be an orthonormal basis of H. Then $\{Te_i \mid i \in I\} \subset H$ is precompact. Hence, in order to show $\|Te_i\| \to 0$, it suffices to show that every convergent sequence in $\{Te_i \mid i \in I\}$ converges to 0. Let $(i_k)_{k \in \mathbb{N}}$ be some sequence in I such that $Te_{i_k} \to \xi \in H$ as $k \to \infty$. Then

$$\|\xi\|^2 = \lim_k \langle Te_{i_k}, \xi \rangle = \lim_k \langle e_{i_k}, T^*\xi \rangle \to 0 \, .$$

So $\|\xi\| = 0$ showing what we had to show.

Now assume that there is no sequence of finite rank projections $(P_n)_{n \in \mathbb{N}}$ such that $\|T - TP_n\| \to 0$. We will show that there is some orthonormal basis $(e_i)_{i \in I}$ such that $\|Te_i\| \not\to 0$. The assumption reformulates to the following statement. There is some $\delta > 0$ such that for all finite rank projections $P \in \mathcal{B}(H)$ we have $\|T - TP\| > \delta$. We construct inductively an orthonormal sequence $(e_n)_{n \in \mathbb{N}}$ such that $\|Te_n\| \geq \delta$. Sine $\|T\| = \|T - T0\| > \delta$, there is a unit vector e_0 such that $\|Te_0\| \geq \delta$. If $e_0, \dots, e_n$ have been constructed, let $K = \mathrm{span}\{e_0, \dots, e_n\}$ and let $P : H \to K$ be the finite rank projection onto K. Then $\|T(1 - P)\| = \|T - TP\| > \delta$ implies the existence of a unit vector $e_{n+1} \in K^\perp$ such that

$\|Te_{n+1}\| \geq \delta$. We obtain the sequence $(e_n)_{n\in\mathbb{N}}$ as claimed above. Completing it to an orthonormal basis $(e_i)_{i\in I}$ of H (with $\mathbb{N} \subset I$), we find that $Te_i \not\to 0$.

If there are finite rank projections $(P_n)_{n\in\mathbb{N}}$ such that $\|T - TP_n\| \to 0$, then $T \in \overline{\mathrm{FR}(H)}$ is a limit of compact operators and hence it is compact itself. This finishes the proof of the proposition. □

It is easy to see that $\mathcal{K}(H) \subset \mathcal{B}(H)$ is a norm-closed ideal. In fact, it is the only norm-closed ideal in $\mathcal{B}(H)$. The following proposition can be proved using measurable functional calculus and the polar decomposition of operators.

Proposition 7.1.4. *Let H be a Hilbert space. Then 0, $\mathcal{K}(H)$ and $\mathcal{B}(H)$ are the only norm closed ideals of $\mathcal{B}(H)$.*

Corollary 7.1.5. *Every non-trivial ideal $I \subset \mathcal{B}(H)$ is a dense subset of the compact operators $\mathcal{K}(H)$.*

Proof. Let $I \trianglelefteq \mathcal{B}(H)$ be some non-trivial idea. Then $\overline{I} \subset \mathcal{B}(H)$ is a closed ideal and hence equals one of the three possibilities 0, $\mathcal{K}(H)$ or $\mathcal{B}(H)$ described by Proposition 7.1.4. If $\overline{I} = 0$, then $I = 0$. Also if $\overline{I} = \mathcal{B}(H)$, then there is an element $T \in I$ such that $\|T - \mathrm{id}_H\| < 1$. Using the von Neumann series, this shows that T is invertible in $\mathcal{B}(H)$ and hence $I = \mathcal{B}(H)$. Since I is non-trivial, it hence follows that $\overline{I} = \mathcal{K}(H)$, which proves the corollary. □

Besides the compact operators there are plenty of non-closed ideals in $\mathcal{B}(H)$. The first examples, that we already know are the operators of finite rank. We are going to consider two more non-closed ideals in $\mathcal{B}(H)$, the trace class operators and the Hilbert-Schmidt operators in Section 7.3.

7.2 The trace on $\mathcal{B}(H)$

In this case we obtain the definition of a trace for operators on an infinite dimensional Hilbert space.

Lemma 7.2.1. *Let H be a Hilbert space, $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ orthonormal bases of H and $T \in \mathcal{B}(H)^+$ a positive operator. Then*

$$\sum_{i\in I} \langle Te_i, e_i\rangle = \sum_{j\in J} \langle Tf_j, f_j\rangle .$$

Proof. Note that for every $i \in I$ and $j \in J$ the term $\langle T^{1/2}e_i, f_j\rangle\langle f_j, T^{1/2}e_i\rangle = |\langle T^{1/2}e_i, f_j\rangle|^2$ is positive. So we may change the order of summation in the following calculation.

$$\begin{aligned}
\sum_{i\in I} \langle Te_i, e_i\rangle &= \sum_{i\in I} \langle T^{1/2}e_i, T^{1/2}e_i\rangle \\
&= \sum_{i\in I, j\in J} \langle T^{1/2}e_i, f_j\rangle\langle f_j, T^{1/2}e_i\rangle \\
&= \sum_{i\in I, j\in J} \langle e_i, T^{1/2}f_j\rangle\langle T^{1/2}f_j, e_i\rangle \\
&= \sum_{j\in J} \langle T^{1/2}f_j, T^{1/2}f_j\rangle \\
&= \sum_{j\in J} \langle Tf_j, f_j\rangle .
\end{aligned}$$

This proves the lemma. □

Thanks to the previous lemma, the next definition is independent of the choice of a basis in a Hilbert space.

Definition 7.2.2. Let H be a Hilbert space. Then $\mathrm{Tr} : \mathcal{B}(H)^+ \to [0, \infty]$ defined by

$$\mathrm{Tr}(T) = \sum_{i \in I} \langle T e_i, e_i \rangle$$

for a some orthonormal basis $(e_i)_{i \in I}$ is called the trace on $\mathcal{B}(H)$.

The trace on a Hilbert space is a natural analogue of the usual trace on a matrix algebra $\mathrm{M}_n(\mathbb{C})$. Indeed, it is easy to observe that if H is a finite dimensional Hilbert space, then Tr is the restriction of the usual trace to the set of positive operators.

The next proposition collects some basic properties of the trace.

Proposition 7.2.3. *Let H be a Hilbert space and $\mathrm{Tr} : \mathcal{B}(H)^+ \to [0, \infty]$ the trace on $\mathcal{B}(H)$.*

- Tr *is additive, that is* $\mathrm{Tr}(T + S) = \mathrm{Tr}(T) + \mathrm{Tr}(S)$ *for all* $T, S \in \mathcal{B}(H)^+$.
- Tr *is positive homogeneous, that is* $\mathrm{Tr}(cT) = c\mathrm{Tr}(T)$ *for all* $T \in \mathcal{B}(H)^+$ *and all* $c \in [0, \infty)$.
- Tr *is finite on finite rank operators, that is* $\mathrm{Tr}(T) < \infty$ *if* $T \in \mathrm{FR}(H)^+$.
- Tr *is tracial in the following senses.*
  - *If* $T \in \mathcal{B}(H)^+$ *and* $U \in \mathcal{U}(H)$, *then* $\mathrm{Tr}(U^*TU) = \mathrm{Tr}(T)$.
  - *If* $T \in \mathcal{B}(H)$, *then* $\mathrm{Tr}(T^*T) = \mathrm{Tr}(TT^*)$.
  - Tr *satisfies* $\mathrm{Tr}(|T|) = \mathrm{Tr}(|T^*|)$ *for all* $T \in \mathcal{B}(H)$.
- Tr *is faithful, that is* $\mathrm{Tr}(T^*T) = 0$ *implies* $T = 0$ *for all* $T \in \mathcal{B}(H)$.

Proof. In order to prove that Tr is additive and positive homogeneous take $T, S \in \mathcal{B}(H)^+$ and $c \in [0, \infty)$ and let $(e_i)_{i \in I}$ be an orthonormal basis of H. Then

$$\mathrm{Tr}(T + cS) = \sum_{i \in I} \langle T + cSe_i, e_i \rangle = \sum_{i \in I} \langle Te_i, e_i \rangle + c\langle Se_i, e_i \rangle = \Big(\sum_{i \in I} \langle Te_i, e_i \rangle\Big) + c\Big(\sum_{i \in I} \langle Se_i, e_i \rangle\Big) = \mathrm{Tr}(T) + c\mathrm{Tr}(S) \,.$$

Next, let $T \in \mathrm{FR}(H)$ be a finite rank operator. Let $(e_i)_{i \in I}$ be an orthonormal basis of H such that $Te_i = 0$ for all but finitely many e_i. Then $\mathrm{Tr}(T) = \sum_{i \in I} \langle Te_i, e_i \rangle$ is a finite sum of positive real numbers. So $\mathrm{Tr}(T) \in [0, \infty)$ is finite.

Let $T \in \mathcal{B}(H)^+$ and $U \in \mathcal{U}(H)$. Fix some orthonormal basis $(e_i)_{i \in I}$ of H and note that $(Ue_i)_{i \in I}$ is an orthonormal basis of H too. Then

$$\mathrm{Tr}(U^*TU) = \sum_{i \in I} \langle U^*TUe_i, e_i \rangle = \sum_{i \in I} \langle TUe_i, Ue_i \rangle = \mathrm{Tr}(T)$$

follows from the independence of Tr of the choice of an orthonormal basis.

Let $T \in \mathcal{B}(H)$ and write $T = V|T|$ for its polar decomposition. Then $TT^* = V|T|^2V^* = VT^*TV^*$ and hence also $V^*TT^*V = T^*T$. Fix an orthonormal basis $(e_i)_{i \in I_0}$ of the support of T and complete it to an orthonormal basis $(e_i)_{i \in I}$ of H, with $I_0 \subset I$. Set $f_i := Ve_i$ for $i \in I_0$ and complete the orthonormal set $(f_i)_{i\in I_0}$ to some orthonormal basis $(f_j)_{j\in J}$ of H with $I_0 \subset J$. Since $T^*f_j = T^*VV^*f_j = 0$ for $j \in J \setminus I_0$, and since Tr is independent of the choice of an orthonormal basis, we obtain

$$\begin{aligned}\mathrm{Tr}(TT^*) &= \sum_{j\in J}\langle TT^*f_j, f_j\rangle \\ &= \sum_{i\in I_0}\langle TT^*f_i, f_i\rangle \\ &= \sum_{i\in I_0}\langle TT^*Ve_i, Ve_i\rangle \\ &= \sum_{i\in I_0}\langle V^*TT^*Ve_i, e_i\rangle \\ &= \sum_{i\in I_0}\langle T^*Te_i, e_i\rangle \\ &= \sum_{i\in I}\langle T^*Te_i, e_i\rangle \\ &= \mathrm{Tr}(T^*T)\,.\end{aligned}$$

Let $T \in \mathcal{B}(H)$ with polar decomposition $T = V|T|$. Put $X = V|T|^{1/2}$. Then $X^*X = |T|^{1/2}V^*V|T|^{1/2} = |T|$, since V is an isometry. We show that $XX^* = |T^*|$, after which the equality $\mathrm{Tr}(|T|) = \mathrm{Tr}(|T^*|)$ follows from traciality of Tr. Since V is an isometry, we obtain a *-homomorphism $\mathrm{Ad}\,V : \mathrm{C}^*(T^*T) \to \mathrm{C}^*(TT^*)$. Since *-homomorphisms are compatible with functional calculus, it follows that

$$XX^* = V|T|V^* = V(T^*T)^{1/2}V^* = (VT^*TV^*)^{1/2} = (TT^*)^{1/2} = |T^*|\,.$$

This shows that Tr is tracial.

Let $T \in \mathcal{B}(H)$ and assume that $\mathrm{Tr}(T^*T) = 0$. Denote by $(e_i)_{i\in I}$ some orthonormal basis of H. Then $0 = \mathrm{Tr}(T^*T)\sum_{i\in I} = \langle T^*Te_i, e_i\rangle = \sum_{i\in I}\|Te_i\|^2$ implies that $Te_i = 0$ for all $i \in I$. Since $\mathrm{span}\{e_i \,|\, i \in I\}$ is dense in H, it follows that $T = 0$. This finishes the proof of the proposition. □

7.3 Trace class and Hilbert-Schmidt operators

We are now going to define the ideal of trace class operators and of Hilbert-Schmidt operators on a Hilbert space.

Definition 7.3.1. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$.

- We call $\|T\|_1 = \mathrm{Tr}(|T|)$ the trace class norm of T. Then T is a trace class operator if $\|T\|_1 < \infty$.
- We call $\|T\|_{\mathcal{HS}} = \mathrm{Tr}(T^*T)$ the Hilbert-Schmidt norm of T. Then T is a Hilbert-Schmidt operator if $\|T\|_{\mathcal{HS}} < \infty$.

We denote the set of trace class operators on H by $\mathcal{T}(H)$ and the set of Hilbert-Schmidt operators by $\mathcal{HS}(H)$.

One important source of Hilbert-Schmidt operators are so called integral kernel operators. We now state a key theorem for our treatment of the representation theory of compact groups (Section 8)

Theorem 7.3.2. *Let X be a locally compact space and μ be a Radon measure space on X. If $k : X \times X \to \mathbb{C}$ is an L^2-kernel, then the integral operator $K : \mathrm{L}^2(X, \mu) \to \mathrm{L}^2(X, \mu)$ associated with k and satisfying $Kf(x) = \int_X k(x, y) f(y) \mathrm{d}\mu(y)$ is well-defined and a Hilbert-Schmidt operator satisfying $\|K\|_{\mathcal{HS}} = \|k\|_2$.*

A proof can be found in [DE14, Proposition 5.3.3.].

We first observe that $\mathcal{T}(H)$ and $\mathcal{HS}(H)$ are closed under conjugation.

Lemma 7.3.3. *For all $T \in \mathcal{B}(H)$ we have $\|T^*\|_1 = \|T\|_1$ and $\|T^*\|_{\mathcal{HS}} = \|T\|_{\mathcal{HS}}$.*

Proof. We use the properties of Tr proven in Proposition 7.2.3, to obtain

$$\|T^*\|_{\mathcal{HS}}^2 = \mathrm{Tr}(TT^*) = \mathrm{Tr}(T^*T) = \|T\|_{\mathcal{HS}}^2$$

and

$$\|T^*\|_1 = \mathrm{Tr}(|T^*|) = \mathrm{Tr}(|T|) = \|T\|_1 .$$

□

In order to prove that trace-class operators and Hilbert-Schmidt operators are ideals of $\mathcal{B}(H)$, we need the following result.

Lemma 7.3.4. *Let $T, S \in \mathcal{B}(H)^+$ such that $T \leq S$. Then $T^{1/2} \leq S^{1/2}$.*

For a proof consult [Mur90, Theorem 2.2.6].

Proposition 7.3.5. *Let $A, B, T \in \mathcal{B}(H)$. Then*

- $\|ATB\|_{\mathcal{HS}} \leq \|A\| \|B\| \|T\|_{\mathcal{HS}}$.
- $\|ATB\|_1 \leq \|A\| \|B\| \|T\|_1$ *and*

Proof. Let $A, B, T \in \mathcal{B}(H)$. We have

$$(AT)^*AT = T^*A^*AT \leq \|A^*A\| T^*T = \|A\|^2 T^*T .$$

This shows that

$$\|AT\|_{\mathcal{HS}}^2 = \mathrm{Tr}((AT)^*AT) \leq \mathrm{Tr}(\|A\|^2 T^*T) = \|A\|^2 \|T\|_{\mathcal{HS}}^2 .$$

By Lemma 7.3.3, we obtain that

$$\|ATB\|_{\mathcal{HS}} \leq \|A\| \|TB\|_{\mathcal{HS}} = \|A\| \|B^*T^*\|_{\mathcal{HS}} \leq \|A\| \|B^*\| \|T^*\|_{\mathcal{HS}} = \|A\| \|B\| \|T\| .$$

In order to obtain similar estimates for the trace-class norm, we apply Lemma 7.3.4 to see that

$$|AT| = ((AT)^*AT)^{1/2} \leq (\|A\|^2 T^*T)^{1/2} = \|A\| |T| .$$

Then also

$$\|AT\|_1 = \mathrm{Tr}(|AT|) \leq \mathrm{Tr}(\|A\| |T|) = \|A\| \|T\|_1 .$$

As before, we obtain

$$\|ATB\|_1 \leq \|A\| \|B\| \|T\|_1$$

follows. This finishes the proof of the proposition. □

Proposition 7.3.6. *$\mathcal{T}(H) \subset \mathcal{HS}(H) \subset \mathcal{K}(H)$ are ideals in $\mathcal{B}(H)$.*

Proof. From Proposition 7.3.5 it is clear that $\mathcal{T}(H)$ and $\mathcal{HS}(H)$ are ideals in $\mathcal{B}(H)$. We only have to show the inclusions $\mathcal{T}(H) \subset \mathcal{HS}(H) \subset \mathcal{K}(H)$. Note that all these sets are spanned by their positive elements. If $T \in \mathcal{T}(H)^+$, then $\|T^{1/2}\|_{\mathcal{HS}} = \|(T^{1/2})^2\|_1 = \|T\|_1 < \infty$ shows that $T^{1/2} \in \mathcal{HS}(H)$. Since $\mathcal{HS}(H)$ is an ideal, it follows that $T = (T^{1/2})^2 \in \mathcal{HS}(H)$. This shows $\mathcal{T}(H) \subset \mathcal{HS}(H)$.

Let now $T \in \mathcal{HS}(H)^+$ and let $(e_i)_{i\in I}$ be some orthonormal basis of H. Then $\infty > \sum_{i\in I}\langle T^*Te_i, e_i\rangle = \sum_{i\in I}\|Te_i\|^2$ shows that $\|Te_i\| \to 0$. So T is compact by Proposition 7.1.3. This shows $\mathcal{HS}(H) \subset \mathcal{K}(H)$ and finishes the proof of the proposition. ☐

7.4 Hilbert-Schmidt operators as a Hilbert space

In this section take another point of view on the Hilbert-Schmidt operators on a Hilbert space H. Not only do they naturally carry the structure of a Hilbert space themselves, but also are they a natural $\mathcal{B}(H)$-bimodule.

Proposition 7.4.1. *Let H be a Hilbert space.*

- *The Hilbert-Schmidt norm on $\mathcal{HS}(H)$ is induced by the inner product $\langle T, S\rangle := \mathrm{Tr}(S^*T)$*
- *$\mathcal{HS}(H)$ is complete and hence a Hilbert space.*

Proof. It is easy to check that $\langle T, S\rangle = \mathrm{Tr}(S^*T)$ defines an inner product on $\mathcal{HS}(H)$ inducing the Hilbert-Schmidt norm. If $0 = \|T\|^2_{\mathcal{HS}} = \mathrm{Tr}(T^*T)$, then $T = 0$ by faithfulness of Tr (Proposition 7.2.3). We show that $\mathcal{HS}(H)$ is complete. Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{HS}(H)$. Then $(\|T_n\xi\|)_{n\in\mathbb{N}}$ is a Cauchy sequence for every $\xi \in H$. We define $T\xi := \lim T_n\xi$. Clearly T is everywhere defined on H and linear. Since for every unit vector $\xi \in H$, we have the estimate

$$\|T\xi\|^2 = \lim_n \|T_n\xi\|^2 \le \lim_n \|T_n\|_{\mathcal{HS}} ,$$

we see that T is a bounded linear operator on H. Fix an orthonormal basis $(e_i)_{i\in I}$ of H. We may apply the Lemma of Fatou and see that T is a Hilbert-Schmidt operator:

$$\|T\|^2_{\mathcal{HS}} = \sum_{i\in I}\|Te_i\|^2 = \sum_{i\in I}\lim_{n\in\mathbb{N}}\|T_ne_i\|^2 \le \liminf_{n\in\mathbb{N}}\sum_{i\in I}\|T_ne_i\|^2 = \liminf_{n\in\mathbb{N}}\|T_n\|_{\mathcal{HS}} .$$

Let $\varepsilon > 0$. Since $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert-Schmidt norm, there is some finite subset $F \subset I$ and some $n_0 \in \mathbb{N}$ such that $\sum_{i\in I\setminus F}\|T_ne_i\|^2 < \varepsilon$ for all $n \ge n_0$. Enlarging F, we may further assume that $\sum_{i\in I\setminus F}\|Te_i\|^2 < \varepsilon$, since T is a Hilbert-Schmidt operator. Making n_0 bigger, we may also assume that $\|(T - T_n)e_i\|^2 < \varepsilon/|F|$ for all $i \in F$. We obtain that for $n \ge n_0$

$$\|T - T_n\|^2_{\mathcal{HS}} = \sum_{i\in I}\|(T - T_n)e_i\|^2 < \sum_{i\in F}\|T - T_n)e_i\|^2 + 2\varepsilon < 3\varepsilon .$$

This shows that $T_n \to T$ in Hilbert-Schmidt norm, so that $\mathcal{HS}(H)$ is complete. It follows that $\mathcal{HS}(H)$ is a Hilbert space. ☐

The next proposition is part of the motivation to introduce Hilbert-Schmidt operators. They provide us with the possibility to obtain a natural bimodule for the bounded operators on a Hilbert space. Let us first fix some notation.

Notation 7.4.2. Let H be a Hilbert space and $\xi, \eta \in H$. The rank one operator $H \ni \zeta \mapsto \langle \zeta, \eta \rangle \xi$ is denoted by $e_{\xi,\eta}$. We admit the relations $e_{\xi,\eta}^* = e_{\eta,\xi}$ and $\mathrm{Tr}(e_{\xi,\eta}) = \langle \xi, \eta \rangle$, which follow from short calculations.

Proposition 7.4.3. *Let H be a Hilbert space.*

- *$H \otimes \overline{H} \to \mathcal{HS}(H) : \xi \otimes \overline{\eta} \mapsto e_{\xi,\eta}$ is a well-defined isomorphism of Hilbert spaces.*
- *$\mathcal{B}(H)$ acts by bounded operators on the left and on the right of $\mathcal{HS}(H)$.*
- *The isomorphism $H \otimes \overline{H} \cong \mathcal{HS}$ intertwines the $\mathcal{B}(H) \otimes_{\mathrm{alg}} \mathcal{B}(H)^{\mathrm{opp}}$ action on $\mathcal{HS}(H)$ with the action $A(\xi \otimes \overline{\eta})B^{\mathrm{opp}} = A\xi \otimes \overline{B^*\eta}$.*

Proof. We first have to show that the assignment $H \otimes \overline{H} \to \mathcal{HS}(H) : \xi \otimes \overline{\eta} \mapsto e_{\xi,\eta}$ is well-defined and isometric. The operator $\sum_{i=1}^n e_{\xi_i,\eta_i}$ is assigned to $\sum_{i=1}^n \xi_i \otimes \overline{\eta_i} \in H \otimes_{\mathrm{alg}} \overline{H}$. We have

$$\begin{aligned}
\| \sum_{i=1}^n e_{\xi_i,\eta_i} \|_{\mathcal{HS}}^2 &= \mathrm{Tr}((\sum_{i=1}^n e_{\xi_i,\eta_i})^* (\sum_{j=1}^n e_{\xi_j,\eta_j})) \\
&= \sum_{i,j=1}^n \mathrm{Tr}(e_{\eta_i,\xi_i} e_{\xi_j,\eta_j}) \\
&= \sum_{i,j=1}^n \langle \xi_i, \xi_j \rangle \mathrm{Tr}(e_{\eta_i,\eta_j}) \\
&= \sum_{i,j=1}^n \langle \xi_i, \xi_j \rangle \langle \eta_i, \eta_j \rangle \\
&= \| \sum_{i=1}^n \xi_i \otimes \overline{\eta_i} \|_{H \otimes \overline{H}}^2
\end{aligned}$$

We obtain an isometric linear map $H \otimes \overline{H} \to \mathcal{HS}(H)$ whose image contains all finite rank operators. So it establishes an isomorphism $H \otimes \overline{H} \cong \mathcal{HS}(H)$.

The fact that $\mathcal{B}(H)$ acts by bounded operators on left and on the right of $\mathcal{HS}$ follows from 7.3.5.

Now let $A, B \in \mathcal{B}(H)$ and $\xi, \eta \in H$. Then

$$(A \otimes B^{\mathrm{op}})(\xi \otimes \overline{\eta}) = A\xi \otimes \overline{B^*\eta} \mapsto e_{A\xi, B^*\eta} = A e_{\xi,\eta} B \,,$$

which shows that the isomorphism described above indeed intertwines the $\mathcal{B}(H) \otimes_{\mathrm{alg}} \mathcal{B}(H)^{\mathrm{op}}$ action on $\mathcal{HS}(H)$ and $H \otimes \overline{H}$. □

We fix the most interesting consequence for us of the previous identification.

Proposition 7.4.4. *Let G be a locally compact group and (π, H) a unitary representation of G. Then $(\mathrm{Ad}\,\pi)(g,h)T = \pi(h)T\pi(g)^*$ defines a representation of $G \times G$ on the Hilbert-Schmidt operators $\mathcal{HS}(H)$.*

Proof. It is clear that the map $\mathrm{Ad}\,\pi : G \times G \to \mathrm{GL}(\mathcal{HS}(H))$ is well-defined. For $T \in \mathcal{HS}(H)$ and $g, h \in G$ we have

$$\|\pi(h)T\pi(g^{-1})\|_{\mathcal{HS}}^2 = \mathrm{Tr}\big(\big(\pi(h)T\pi(g^{-1})\big)^* \pi(h)T\pi(g^{-1})\big) = \mathrm{Tr}(T^*T) = \|T\|_{\mathcal{HS}}^2 \,.$$

So $\operatorname{Ad}\pi : G \times G \to \mathcal{U}(\mathcal{HS}(H))$. We have to check that it is continuous. Since finite rank operators are dense in $\mathcal{HS}(H)$ with respect to the Hilbert-Schmidt norm, it suffices to check continuity on these. Proposition 7.4.3 tells us that there is an isomorphism $\varphi : \mathcal{HS}(H) \cong H \otimes \overline{H}$ satisfying $e_{\xi,\eta} = \xi \otimes \overline{\eta}$ for all $\xi, \eta \in H$. In particular, φ identifies the subspace of finite rank operators with the algebraic tensor product $H \otimes_{\mathrm{alg}} \overline{H}$. Further,

$$\varphi((\operatorname{Ad}\pi)(g,h)(e_{\xi,\eta})) = \varphi(e_{\pi(g)\xi,\pi(h)\eta}) = \pi(g)\xi \otimes \overline{\pi(h)}\overline{\eta}\,,$$

showing that $\operatorname{Ad}\pi$ is indeed pointwise continuous on finite rank operators. □

8 Compact groups: Peter-Weyl Theory and characters

In this section we will investigate different aspects of the unitary representation theory of compact groups. Let us start by some facts on their Haar measure.

Proposition 8.0.1. *Let K be a compact group. Then the Haar measure on K is finite and the modular function of K is trivial, that is $\Delta_K \equiv 1$.*

Proof. Assuming that G is compact, we take an open subset $U \subset G$ of finite Haar measure, which exists thanks to the assumption of local finiteness on Haar measures. Since G is compact, there are finitely many elements $g_1, \dots, g_n \in G$ such that $G = \bigcup_{i=1}^n g_i U$. So $\mu(G) \leq \sum_{i=1}^n \mu(g_i U) = n\mu(U) < \infty$, where μ is the Haar measure of G. This shows that μ is finite.

Let μ be the left Haar probability measure of K. Since $\mathbb{1}_K(xy) = 1$ for all $x, y \in K$, we obtain for $x \in K$

$$\Delta(x) = \Delta(x) \int_K \mathbb{1}_K(y) \mathrm{d}y = \int_K \mathbb{1}_K(yx) \mathrm{d}y = 1\,.$$

We then conclude $\Delta(x) = 1$. ☐

From now on we will normalise the Haar measure on a compact group to be a probability measure.

Although not right away necessary, it is advantageous to take note of the following proposition, which expresses the fact that unitary representation theory of a compact group covers actually a wider range of representations than expected naively.

Proposition 8.0.2. *Let K be a compact group and let $\pi : K \to \mathrm{GL}(H)$ be a not necessarily unitary representation on a Hilbert space V. Then π is unitarisable.*

Proof. Let $(\cdot,\cdot)$ be the scalar product of H. We define a function on $H \times H$ by $\langle \xi, \eta \rangle := \int_K (\pi(x)\xi, \pi(x)\eta) \mathrm{d}x$ and claim that $\langle \cdot,\cdot \rangle$ is a K-invariant scalar product, which induces the topology of H. This will show that K is unitarisable.

From linearity of the integral it follows that $\langle \cdot,\cdot \rangle$ is a sesquilinear form on H. Since the integral of a non-negative real valued function is non-negative, it follows that $\langle \cdot,\cdot \rangle$ is positive semi-definite. Further, the map $K \ni x \mapsto (\pi(x)\xi, \pi(x)\xi)$ is continuous for every ξ in H, because π is pointwise continuous. So if $\langle \xi, \xi \rangle = 0$, then $(\pi(x)\xi, \pi(x)\xi) = 0$ for all $x \in K$. It follows that $\xi = 0$. We showed that $\langle \cdot,\cdot \rangle$ is a scalar product on H.

For $x \in K$, $\xi, \eta \in H$, right invariance of the Haar measure shows that

$$\langle \pi(x)\xi, \pi(x)\eta \rangle = \int_G (\pi(y)\pi(x)\xi, \pi(y)\pi(x)\eta) \mathrm{d}y = \int_G (\pi(y)\xi, \pi(y)\eta) \mathrm{d}y = \langle \xi, \eta \rangle\,.$$

So $\langle \cdot,\cdot \rangle$ is K-invariant.

Since $K \to \mathrm{GL}(H)$ is a representation, the function $x \mapsto \|\pi(x)\|$ is locally bounded and hence bounded, because K is compact. So there is $C > 0$ such that $\|\pi(x)\| \leq C$ for all $x \in K$. Then $\|\pi(x)\| = \|\pi(x^{-1})^{-1}\| \geq 1/C$ follows for all $x \in K$. Thus $\|\pi(K)\| \subset [1/C, C]$. It follows that for $\xi \in H$

$$\|\xi\|_{\langle \cdot,\cdot \rangle}^2 = \int_G (\pi(x)\xi, \pi(x)\xi) \mathrm{d}x = \int_G \|\pi(x)\xi\|_{(\cdot,\cdot)}^2 \mathrm{d}x \in \|\xi\|_{(\cdot,\cdot)}^2 [1/C, C]\,.$$

This shows that the norms induced by $(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle$ are equivalent. So the latter induces the topology on H. This finishes the proof of the proposition. ☐

8.1 Peter-Weyl theory

Peter-Weyl theory describes irreducible unitary representations of a compact group by decomposing its regular representation into a direct sum of finite dimensional irreducible representations exhausting all irreducible representations of the compact group.

Lemma 8.1.1. *Let G be a unimodular locally compact group and $f, g \in \mathrm{L}^2(G)$. Then $f * g$ is well-defined everywhere and continuous.*

Proof. As G is unimodular, we have $g^* \in \mathrm{L}^2(G)$. Hence, the scalar product $\langle f, \mathrm{L}_x g^* \rangle$ makes sense for every $x \in G$. It is continuous in x by Lemma 4.3.1. Since

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) \mathrm{d}y = \int_G f(y) \overline{g^*(x^{-1}y)} \mathrm{d}y = \langle f, \mathrm{L}_x g^* \rangle$$

this finishes the proof of the lemma. □

Lemma 8.1.2. *Let G be a locally compact group. If $\xi \in \mathrm{L}^2(G)$ is a λ_G-invariant vector, then ξ is a multiple of the constant function.*

Proof. Let $\xi \in \mathrm{L}^2(G)$ be a λ_G-invariant vector and chose a direct net $(f_u)_{U \in \mathcal{U}}$ for G consisting of elements from $(\mathrm{L}^2 \cap \mathrm{L}^1)(G)$. By Lemma 8.1.1, we have $\rho(\mathbb{f}_U)\xi \in \mathrm{L}^2(G) \cap \mathrm{C}_0(G)$ for every $U \in \mathcal{U}$. Note that $\rho(f_U)\xi$ is λ_G-invariant, since λ_G and ρ_G commute by Proposition 4.3.7. So $\rho(f_U)\xi \in \mathbb{C}\mathbb{1}_G$ by continuity and left G-invariance. Letting $U \downarrow \{e\}$ run through $\mathcal{U}$, Lemma 6.2.4 applies to show that $\rho(f_U)$ converges to $\mathrm{id}_{\mathrm{L}^2(G)}$ strongly. We conclude that $\xi = \lim_U \rho(f_U)\xi \in \mathbb{C}\mathbb{1}_G$. □

Proposition 8.1.3. *Let G be a locally compact group. The following statements are equivalent.*

- *G is compact.*
- *The Haar measure of G is finite.*
- *$\mathbb{1}_G \in \mathrm{L}^2(G)$.*
- *$1_G \leq \lambda_G$.*
- *$1_G \leq \lambda_G$ with multiplicity 1.*

Proof. We first show that G is compact if and only if its Haar measure is finite. The forward implication was already proven in Proposition 8.0.1. Assume that G is not compact and let $O \subset G$ be a compact set with non-empty interior. For $g, h \in G$ we have $gO \cap hO \neq \emptyset$ if and only if $g \in hO \cdot O^{-1}$. Since G is not compact, it is not the union of finitely many translates of $O \cdot O^{-1}$. Hence we find a sequence of elements $(g_i)_{i \in \mathbb{N}}$ such that $g_i O \cap g_j O = \emptyset$ for all $i \neq j$. We infer that $\mu(G) \geq \mu(\bigcup_{i \in \mathbb{N}} g_i O) = \sum_{i \in \mathbb{N}} \mu(g_i O) = \infty$.

We show that the remaining statements of the proposition are equivalent to G having finite Haar measure. If the Haar measure of G is finite, then $\mathbb{1}_G$ is square integrable and hence $\mathbb{1}_G \in \mathrm{L}^2(G)$. Assuming that $\mathbb{1}_G \in \mathrm{L}^2(G)$, there is a non-zero G-invariant vector in $\mathrm{L}^2(G)$, which shows that $1_G \leq \lambda_G$. Next assuming that $1_G \leq \lambda_G$ and $\xi \in \mathrm{L}^2(G)$ is λ_G-invariant, we find that $\xi \in \mathbb{C}\mathbb{1}_G$ by Lemma 8.1.2. So 1_G has multiplicity one in λ_G. Finally, we assume that $1_g \leq \lambda_G$ (with multiplicity one). Then there is a non-zero λ_G-invariant vector in $\mathrm{L}^2(G)$. By Lemma 8.1.2 it is a multiple of the constant function, so that we have $\mathbb{1}_G \in \mathrm{L}^2(G)$. This shows that the Haar measure of G is finite. □

Lemma 8.1.4. *Let K be a compact group and π an irreducible unitary representation of K. Then π is contained in the left-regular representation of K. If π is finite dimensional, then its multiplicity in λ is at most $\dim \pi$.*

Proof. Since K is compact, we have $1 \leq \lambda$ by Proposition 8.1.3. So Fell's absorption principle (Proposition 4.3.5) shows that

$$\pi \cong \pi \otimes 1 \leq \pi \otimes \lambda \cong (\dim \pi)\lambda \,.$$

Since π is irreducible, this shows that $\pi \leq \lambda$.

Assume that π is finite dimensional. Denote by m_π^λ the multiplicity of π in λ. If $\overline{\pi}$ denotes the conjugate representation of π, then Proposition 3.1.10 says that $1 \leq \overline{\pi} \otimes \pi$. So $m_\pi^\lambda \leq m_1^{\overline{\pi} \otimes \lambda}$, where the latter denotes the multiplicity of 1 in $\overline{\pi} \otimes \lambda$. Further,

$$\overline{\pi} \otimes \lambda \cong (\dim \overline{\pi})\lambda \cong (\dim \pi)\lambda$$

in combination with Proposition 8.1.3 shows that $m_1^{\overline{\pi} \otimes \lambda} \leq \dim \pi m_1^\lambda = \dim \pi$. This finishes the proof of the lemma. □

Before we prove the next lemma, we recall integral operators. If X is a locally compact space with a σ-finite Radon measure, then every L^2-kernel $F : X \times X \to \mathbb{C}$ gives rise to a Hilbert-Schmidt integral operator $K_F : \mathrm{L}^2(X) \to \mathrm{L}^2(X)$ defined by $(K_F \xi)(x) = \int_X f(y) F(x,y) \mathrm{d}y$.

Lemma 8.1.5. *Let K be a compact group and $f \in \mathrm{L}^2(K)$. Then $\lambda(f)$ and $\rho(f)$ and are Hilbert-Schmidt operators.*

Proof. First observe that $\mathrm{L}^2(K) \subset \mathrm{L}^1(K)$, since the Haar measure of K is finite. So the statement of the lemma makes sense. Since $\lambda \cong \rho$ by Proposition 4.3.3, it suffices to show that $\lambda(f)$ is Hilbert-Schmidt for all $f \in \mathrm{L}^2(K)$. Consider the continuous kernel $F : K \times K \to \mathbb{C}$ defined as $F(x,y) = f(yx^{-1})$. Since F is continuous on the compact set $K \times K$, it is 2-integrable. Hence the integral operator K_F is Hilbert-Schmidt on $\mathrm{L}^2(K)$. We next calculate with $g, h \in \mathrm{L}^2(K)$.

$$\begin{aligned}
\langle \lambda(f) g, h \rangle &= \int_K \langle f(x) \lambda_x g, h \rangle \mathrm{d}x \\
&= \int_K \int_K f(x) (\lambda_x g)(y) \overline{h(y)} \mathrm{d}y \mathrm{d}x \\
&= \int_K \int_K f(x) g(x^{-1} y) \overline{h(y)} \mathrm{d}y \mathrm{d}x \\
&= \int_K \int_K f(x) g(x^{-1} y) \overline{h(y)} \mathrm{d}x \mathrm{d}y \\
&= \int_K \int_K f(yx) g(x^{-1}) \overline{h(y)} \mathrm{d}x \mathrm{d}y \\
&= \int_K \int_K f(yx^{-1}) g(x) \overline{h(y)} \mathrm{d}x \mathrm{d}y \qquad \text{(unimodularity)} \\
&= \int_K (K_F g)(y) \overline{h(y)} \mathrm{d}y \\
&= \langle K_F g, h \rangle \,.
\end{aligned}$$

It follows that $\lambda(f) = K_F$ is Hilbert-Schmidt. □

Remark 8.1.6. It follows immediately from Proposition 3.1.4 that a finite dimensional unitary representation of a topological group is the direct sum of irreducible representations.

We next come to one important intermediate step to the Peter-Weyl theorem.

Lemma 8.1.7. *The regular representation of a compact group is a direct sum of finite dimensional irreducible unitary representations.*

Proof. Let K be a compact group. In view of Remark 8.1.6, it suffices to show that the regular representation of K is a direct sum of finite dimensional representations. Let $H := \{\xi \in \mathrm{L}^2(K) \mid \operatorname{span} \lambda(K)\xi \text{ is fin. dim}\}$. We have to show that $H = \mathrm{L}^2(K)$. We assume that this is not the case and we will deduce a contradiction. First note that H is $\lambda \times \rho$-invariant. Next, if $f \in H^\perp$, then $\lambda(f_U)f \to f$ where $(f_U)_U$ is a Dirac net in $\mathrm{L}^2(K)$. Since Lemma 8.1.1 says that $\lambda(f_U)f \in \mathrm{C}(K)$ it follows that $H^\perp \cap \mathrm{C}(K) \neq 0$. Take $f \in H^\perp \cap \mathrm{C}(K)$. Then $\rho(\overline{f})$ is a Hilbert-Schmidt operator by Lemma 8.1.5 and hence $T = \rho(\overline{f})^* \rho(\overline{f})$ is a positive compact operator in $\rho(\mathrm{L}^1(K))$. We have

$$(\rho(\overline{f})f)(x) = \int_K \overline{f(y)} f(xy) \mathrm{d}y = \int_K \overline{f(y) f^*(y^{-1}x^{-1})} \mathrm{d}y = \overline{(f * f^*)(x^{-1})},$$

showing that $\langle Tf, f\rangle = \langle \rho(\overline{f})f, \rho(\overline{f})f\rangle \neq 0$. So $TH^\perp \neq 0$.

Theorem 7.1.2 provides us with spectral projections E_ν, $\nu \in \sigma(T)$ of the positive compact operator T. Then E_ν is a finite rank projection for every $\nu \neq 0$. Since $TH^\perp \neq 0$, there is some $\nu \neq 0$ such that $\mathrm{E}_\nu H^\perp \neq 0$. Further, $T \in \rho(\mathrm{L}^1(K))$ implies that $\mathrm{E}_\nu \in \mathrm{C}^*(\rho(\mathrm{L}^1(K)))$, which shows that $\mathrm{E}_\nu H^\perp \subset H^\perp$ is a λ-invariant subspace. We thus found a non-zero finite dimensional λ-invariant subspace inside $H^\perp$, which contradicts the definition of H. This finishes the proof of the lemma. □

Lemma 8.1.7 allows us to obtain the following statement on complete reducibility of unitary representations of a compact group.

Proposition 8.1.8. *Let K be a compact group. Then every unitary representation of K is completely reducible.*

Proof. Let π be a unitary representation of K. Then $\pi \leq (\dim \pi)\lambda_K$ by Fells absorption principle. So Lemma 8.1.7 says that π is a direct sum of irreducible representations. □

Summarising the content of this section up to this point, Lemmas 8.1.4 and 8.1.7 show that every irreducible representation of a compact group is finite and its regular representation is a direct sum of these irreducible representations each appearing with multiplicity bounded by its dimension. Our next aim is to show that this multiplicity actually equals the dimension of the irreducible representation. We are going to do so, by exhibiting sufficiently many functions in L^2 which transform the same way as vectors in the given irreducible representation. The functions associated with an irreducible representation are so called matrix coefficients in the sense of the following definition. Note that every matrix coefficient is a continuous function.

Definition 8.1.9. Let G be a locally compact group and (π, H) a unitary representation of G. Every function $f(g) = \langle \pi(g)\xi, \eta\rangle$ for $\xi, \eta \in H$ is called a matrix coefficient of π.

Lemma 8.1.10. *Let K be a compact group and (π, H) an irreducible representation of K. Let $(e_i)_{i=1}^{\dim \pi}$ be an orthonormal basis for H. Define the matrix coefficients*

$$f_{ij}(g) := \langle \pi(g)e_i, e_j\rangle .$$

Then the orthogonality relations

$$\langle \lambda_g f_{ij}, f_{kl}\rangle = \delta_{ik}\frac{1}{\dim\pi}\langle \overline{\pi}(g)\overline{e_j}, \overline{e_l}\rangle$$
$$\langle \rho_g f_{ij}, f_{kl}\rangle = \delta_{jl}\frac{1}{\dim\pi}\langle \pi(g)e_i, e_k\rangle$$

hold for all $1 \leq i,j,k,l \leq \dim\pi$ and all $g \in K$. In particular, the following statements are true.

- *The elements $(\sqrt{\dim\pi} f_{ij})_{i,j=1}^{\dim\pi}$ form an orthonormal system in $\mathrm{L}^2(K)$.*
- *For every $1 \leq i \leq \dim\pi$, the space $\mathrm{span}\{f_{ij} \mid 1 \leq j \leq \dim\pi\}$ is a λ-invariant subspace of $\mathrm{L}^2(K)$. The restriction of λ to this subspace is isomorphic to $\overline{\pi}$.*
- *For every $1 \leq j \leq \dim\pi$, the space $\mathrm{span}\{f_{ij} \mid 1 \leq i \leq \dim\pi\}$ is a ρ-invariant subspace of $\mathrm{L}^2(K)$. The restriction of ρ to this subspace is isomorphic to π.*

Proof. It suffices to show the two equalities

$$\langle \lambda_g f_{ij}, f_{kl}\rangle = \delta_{ik}\frac{1}{\dim\pi}\langle \overline{\pi}(g)\overline{e_j}, \overline{e_l}\rangle \tag{8.1}$$
$$\langle \rho_g f_{ij}, f_{kl}\rangle = \delta_{jl}\frac{1}{\dim\pi}\langle \pi(g)e_i, e_k\rangle \tag{8.2}$$

for all $1 \leq i,j,k,l \leq \dim\pi$ and all $g \in K$.

For $1 \leq i,j \leq \dim\pi$ consider the rank-one operator $T_{ij} : H \to H : \xi \mapsto \langle \xi, e_i\rangle e_j$. Putting $S_{ij} := \int_K \pi(k^{-1})T_{ij}\pi(k)\mathrm{d}k$, we obtain an intertwiner S_{ij} from π to π, meaning that $S_{ij}\pi(k) = \pi(k)S_{ij}$ for all $k \in K$. We hence have $S_{ij} = c_{ij}\mathrm{id}_H$ for some scalar c_{ij} by the Lemma of Schur (Corollary 3.2.5). We have

$$\begin{aligned}
c_{ij}(\dim\pi) &= \mathrm{tr}(S_{ij}) \\
&= \mathrm{tr}(\int_K \pi(k^{-1}T_{ij}\pi(k)\mathrm{d}k) \\
&= \int_K \mathrm{tr}(\pi(k^{-1})T_{ij}\pi(k))\mathrm{d}k \\
&= \int_K \mathrm{tr}(T_{ij})\mathrm{d}k \\
&= \delta_{ij}\,.
\end{aligned}$$

It follows that $S_{ij} = \delta_{ij}\frac{1}{\dim\pi}\mathrm{id}_H$ for all $1 \leq i,j \leq \dim\pi$.

We fist show that

$$\langle \lambda_g f_{ij}, f_{kl}\rangle = \delta_{ik}\frac{1}{\dim\pi}\langle \overline{\pi}(g)\overline{e_j}, \overline{e_l}\rangle\,.$$

For variables $1 \leq i,l,r,k \leq \dim\pi$ we have

$$\begin{aligned}
\int_K \langle \pi(h)e_i, e_r\rangle\langle \pi(h^{-1})e_l, e_k\rangle \mathrm{d}h &= \int_K \langle \pi(h^{-1})T_{rl}\pi(h)e_i, e_k\rangle \mathrm{d}h \\
&= \langle \Big(\int_K \pi(h^{-1})T_{rl}\pi(h)\mathrm{d}h\Big)e_i, e_k\rangle \\
&= \langle S_{rl}e_i, e_k\rangle \\
&= \frac{1}{\dim\pi}\delta_{rl}\delta_{ik}\,.
\end{aligned}$$

So we obtain for $1 \leq i,j,k,l \leq \dim \pi$ and $g \in K$ that

$$\begin{aligned}
\langle \lambda_g f_{ij}, f_{kl} \rangle &= \int_K f_{ij}(g^{-1}h)\overline{f_{kl}(h)} \mathrm{d}h \\
&= \int_K \langle \pi(g^{-1}h)e_i, e_j \rangle \overline{\langle \pi(h)e_k, e_l \rangle} \mathrm{d}h \\
&= \int_K \langle \pi(h)e_i, \pi(g)e_j \rangle \langle \pi(h^{-1})e_l, e_k \rangle \mathrm{d}h \\
&= \sum_{r=1}^{\dim \pi} \int_K \langle \pi(h)e_i, e_r \rangle \langle e_r, \pi(g)e_j \rangle \langle \pi(h^{-1})e_l, e_k \rangle \mathrm{d}h \\
&= \sum_{r=1}^{\dim \pi} \langle e_r, \pi(g)e_j \rangle \Big(\int_K \langle \pi(h)e_i, e_r \rangle \langle \pi(h^{-1})e_l, e_k \rangle \mathrm{d}h \Big) \\
&= \sum_{r=1}^{\dim \pi} \langle e_r, \pi(g)e_j \rangle \frac{1}{\dim \pi} \delta_{rl}\delta_{ik} \\
&= \frac{1}{\dim \pi} \delta_{ik} \langle e_l, \pi(g)e_j \rangle \\
&= \frac{1}{\dim \pi} \delta_{ik} \langle \overline{\pi}(g)\overline{e_j}, \overline{e_l} \rangle \,.
\end{aligned}$$

This shows the first formula (8.1).

In order to prove the second equation (8.2), we first take indices $1 \leq j,k,r,l \leq \dim \pi$ and obtain

$$\begin{aligned}
\int_K \langle e_r, \pi(h^{-1})e_j \rangle \langle \pi(h)e_k, e_l \rangle \mathrm{d}h &= \int_K \langle e_r, \pi(h^{-1})T_{jl}\pi(h)e_k \rangle \mathrm{d}h \\
&= \langle e_r, \Big(\int_K \pi(h^{-1})T_{jl}\pi(h) \mathrm{d}h \Big) e_k \rangle \\
&= \langle e_r, S_{jl}e_k \rangle \\
&= \frac{1}{\dim \pi} \delta_{jl}\delta_{rk} \,.
\end{aligned}$$

Using this equation, we obtain for $1 \leq i,j,k,l \leq \dim \pi$ and for $g \in K$

$$\begin{aligned}
\langle \rho_g f_{ij}, f_{kl} \rangle &= \int_K f_{ij}(kg)\overline{f_{kl}(h)} \mathrm{d}h \\
&= \int \langle \pi(hg)e_i, e_j \rangle \overline{\langle \pi(h)e_k, e_l \rangle} \mathrm{d}h \\
&= \int_K \langle \pi(g)e_i, \pi(h^{-1})e_j \rangle \langle e_l, \pi(h)e_k \rangle \mathrm{d}h \\
&= \sum_{r=1}^{\dim \pi} \langle \pi(g)e_i, e_r \rangle \Big(\int_K \langle e_r, \pi(h^{-1})e_j \rangle \langle e_l, \pi(h)e_k \rangle \mathrm{d}h \Big) \\
&= \sum_{r=1}^{\dim \pi} \langle \pi(g)e_i, e_r \rangle \frac{1}{\dim \pi} \delta_{jl}\delta_{rk} \\
&= \frac{1}{\dim \pi} \delta_{lj} \langle \pi(g)e_i, e_k \rangle \,.
\end{aligned}$$

This shows the second equation (8.2) and finishes the proof of the lemma. □

We summarise the Lemmas 8.1.4, 8.1.7 and 8.1.10 to obtain the Peter-Weyl theorem.

Theorem 8.1.11. *Let K be a compact group. For $\pi \in \hat{K}$ denote by $(f_{ij}^\pi)_{i,j=1}^{\dim \pi}$ be the matrix coefficients associated with an orthonormal basis of the underlying Hilbert space of π. Then $(\sqrt{\dim \pi} f_{ij}^\pi)_{\pi \in \hat{K}, i,j \in \{1,\dots,\dim \pi\}}$ is an orthonormal basis for $\mathrm{L}^2(K)$.*

We are next going to give an alternative formulation of the Peter-Weyl theorem, which is less explicit in terms of the choice of an orthonormal basis of $\mathrm{L}^2(K)$, but is more strongly emphasising the structure of the representation of $K \times K$ on $\mathrm{L}^2(K)$ which is only implicitly described in Lemma 8.1.10 and Theorem 8.1.11. Let us start with a base-free formulation of the orthogonality relations proven in Lemma 8.1.10.

Lemma 8.1.12. *Let K be a compact group and (π, H) an irreducible representation of K. For $\xi, \eta \in H$ denote the associated matrix coefficient by $f_{\xi,\eta}$. Then*

$$\langle f_{\xi_1,\eta_1}, f_{\xi_2,\eta_2} \rangle_{\mathrm{L}^2(K)} = \frac{1}{\dim \pi} \langle \xi_1, \xi_2 \rangle \langle \eta_2, \eta_1 \rangle ,$$

for all $\xi_1, \xi_2, \eta_1, \eta_2 \in H$.

Proof. Let $(e_i)_i$ be a basis of H and let $\xi_1, \xi_2, \eta_1, \eta_2 \in H$. We calculate, using the orthogonality relations proven in Lemma 8.1.10.

$$\begin{aligned}
\langle f_{\xi_1,\eta_1}, f_{\xi_2,\eta_2} \rangle_{\mathrm{L}^2(K)} &= \int_K f_{\xi_1,\eta_1}(g) \overline{f_{\xi_2,\eta_2}(g)} \mathrm{d}dg \\
&= \int_K \langle \pi(g)\xi_1, \eta_1 \rangle \overline{\langle \pi(g)\xi_2, \eta_2 \rangle} \mathrm{d}g \\
&= \sum_{i,j} \int_K \langle \pi(g)\xi_1, e_i \rangle \langle e_i, \eta_1 \rangle \langle \eta_2, e_j \rangle \langle e_j, \pi(g)\xi_2 \rangle \mathrm{d}g \\
&= \sum_{i,j,k,l} \int_K \langle \xi_1, e_k \rangle \langle \pi(g)e_k, e_i \rangle \langle e_i, \eta_1 \rangle \langle \eta_2, e_j \rangle \langle e_j, \pi(g)e_l \rangle \langle e_l, \xi_2 \rangle \mathrm{d}g \\
&= \sum_{i,j,k,l} \langle \xi_1, e_k \rangle \langle e_i, \eta_1 \rangle \langle \eta_2, e_j \rangle \langle e_l, \xi_2 \rangle \langle f_{ki}, f_{lj} \rangle_{\mathrm{L}^2(K)} \\
&= \frac{1}{\dim \pi} \sum_{i,j,k,l} \delta_{kl} \delta_{ij} \langle \xi_1, e_k \rangle \langle e_i, \eta_1 \rangle \langle \eta_2, e_j \rangle \langle e_l, \xi_2 \rangle \\
&= \frac{1}{\dim \pi} \sum_{i,k} \langle \xi_1, e_k \rangle \langle e_k, \xi_2 \rangle \langle \eta_2, e_i \rangle \langle e_i, \eta_1 \rangle \\
&= \frac{1}{\dim \pi} \langle \xi_1, \xi_2 \rangle \langle \eta_2, \eta_1 \rangle .
\end{aligned}$$

□

We can now establish a base-free formulation of the remaining statements of Lemma 8.1.10.

Lemma 8.1.13. *Let K be a compact group and (π, H) an irreducible unitary representation of K. The map $V : \mathcal{HS}(H) \to \mathrm{L}^2(K)$ satisfying $V(T)(g) = \sqrt{\dim \pi} \mathrm{Tr}(\pi(g)T)$ is a well-defined isometric intertwiner from $\mathrm{Ad}\, \pi$ to $\lambda \times \rho$.*

Proof. Since $\mathcal{HS}(H)$ is densely spanned by rank-one operators, it suffices to check that V preserves the scalar product on such elements. Note that for all $\xi, \eta \in H$ we have

$$V(e_{\xi,\eta})(g) = \sqrt{\dim \pi} \mathrm{Tr}(\pi(g) e_{\xi,\eta}) = \sqrt{\dim \pi} \mathrm{Tr}(e_{\pi(g)\xi,\eta}) = \sqrt{\dim \pi} \langle \pi(g)\xi, \eta \rangle = \sqrt{\dim \pi} f_{\xi,\eta}(g) ,$$

where the last term is the evaluation of a matrix coefficient. Using Lemmas 7.4.3 and 8.1.12, we obtain for $\xi_1, \xi_2, \eta_1, \eta_2 \in H$ that

$$\begin{aligned}\langle V e_{\xi_1,\eta_1}, V e_{\xi_2,\eta_2}\rangle_{L^2(K)} &= \dim \pi \langle f_{\xi_1,\eta_1}, f_{\xi_2,\eta_2}\rangle_{L^2(K)} \\ &= \langle \xi_1, \xi_2\rangle\langle \eta_2, \eta_1\rangle \\ &= \langle \xi_1, \xi_2\rangle\langle \overline{\eta_1}, \overline{\eta_2}\rangle \\ &= \langle e_{\xi_1,\eta_1}, e_{\xi_2,\eta_2}\rangle_{\mathcal{HS}(H)} .\end{aligned}$$

This shows that V is a well-defined isometry. We check that V is an intertwiner from $\mathrm{Ad}\,\pi$ to $\lambda \times \rho$. Again it suffices to check this on a generating set, that is $V(\mathrm{Ad}\,\pi)(g,h)e_{\xi,\eta} = (\lambda \times \rho)(g,h)Ve_{\xi,\eta}$ for all $g, h \in K$ and $\xi, \eta \in H$. For $k \in K$ we have

$$\begin{aligned}(V(\mathrm{Ad}\,\pi)(g,h)e_{\xi,\eta})(k) &= \sqrt{\dim \pi}\,\mathrm{Tr}(\pi(k)\pi(h)e_{\xi,\eta}\pi(g)^*) \\ &= \sqrt{\dim \pi}\,\mathrm{Tr}(e_{\pi(kh)\xi,\pi(g)\eta}) \\ &= \sqrt{\dim \pi}\,\mathrm{Tr}(e_{\pi(g^{-1}kh)\xi,\eta}) \\ &= ((\lambda \times \rho)(g,h)Ve_{\xi,\eta})(k) .\end{aligned}$$

This finishes the proof of the lemma. □

We can now give a second formulation of the Peter-Weyl theorem.

Theorem 8.1.14. *Let K be a compact group. For every $\pi \in \hat{K}$ denote by $V_\pi : \mathcal{HS}(H_\pi) \to L^2(K)$ the isometry satisfying $V_\pi(T)(g) = \sqrt{\dim \pi}\,\mathrm{Tr}(\pi(g)T)$ for all $g \in K$. Then*

$$\bigoplus_{\pi \in \hat{K}} V_\pi : \ell^2\text{-}\bigoplus_{\pi \in \hat{K}} \mathcal{HS}(H_\pi) \to L^2(K)$$

is a unitary equivalence between $\bigoplus_{\pi \in \hat{K}} \mathrm{Ad}\,\pi$ and $\lambda \times \rho$.

Proof. For $\pi \in \hat{K}$, $\xi, \eta \in H$ and $g \in K$, we have $e_{\xi,\eta}\pi(g)^* = e_{\xi,\pi(g)\eta}$. So the definition of $\mathrm{Ad}\,\pi$ and Lemma 8.1.13 show that $\lambda|_{\mathrm{im}\,V_\pi} \cong (\dim \pi)\pi$. In particular, the images of V_π, $\pi \in \hat{K}$ are pairwise orthogonal. Hence $V = \bigoplus_{\pi \in \hat{K}} V_\pi$ is an isometric intertwiner from $\bigoplus_{\pi \in \hat{K}} \mathrm{Ad}\,\pi$ to $\lambda \times \rho$. We have $\lambda|_{\mathrm{im}\,V} \cong \bigoplus_{\pi \in \hat{K}} (\dim \pi)\pi$ so that Theorem 8.1.11 implies surjectivity of V. □

8.2 Character theory

Character theory allows us to study unitary representations of compact groups by means of certain functions.

Definition 8.2.1. Let G be topological group and (π, H) a finite dimensional unitary representation of G. Then $\chi_\pi(g) := \mathrm{Tr}(\pi(g))$ is called the character of π.

Let us right away fix the observation that a character only depends on the isomorphism class of a unitary representation.

Lemma 8.2.2. *Let G be a topological group. The character of a finite dimensional unitary representation π of G only depends on the isomorphism class of π.*

Proof. Let $\pi \cong \nu$ be isomorphic finite dimensional unitary representations of G. Then there is an intertwiner $U : H_\pi \to H_\nu$. We have

$$\chi_\nu(x) = \mathrm{Tr}_{H_\nu}(\nu(x)) = \mathrm{Tr}_{H_\nu}(U\pi(x)U^*) = \mathrm{Tr}_{H_\pi}(\pi(x)) = \chi_\pi(x)\,,$$

by the uniqueness of the trace on $\mathcal{B}(H_\pi)$. This finishes the proof of the lemma. ☐

Remark 8.2.3. If G is a topological group and π a finite dimensional unitary representation of G, then χ_π is a conjugation invariant function on G. This follows from the properties of the trace. For all $x, y \in G$ we have

$$\chi_\pi(xy) = \mathrm{Tr}(\pi(xy)) = \mathrm{Tr}(\pi(x)\pi(y)) = \mathrm{Tr}(\pi(y)\pi(x)) = \chi_\pi(yx)\,.$$

For compact groups, Peter-Weyl theory allows us to recover the isomorphism class of a unitary representation from its character alone, which establishes a powerful correspondence between unitary representations and certain functions on the compact group.

Proposition 8.2.4. *Let K be a compact group.*

- *Let $\pi, \nu \in \hat{K}$ be irreducible unitary representations of K. Then $\langle \chi_\pi, \chi_\nu \rangle_{\mathrm{L}^2(K)} = \delta_{\pi,\nu}$.*
- *The isomorphism class of a finite dimensional unitary representation π of K is uniquely determined by its character.*

Proof. Let $\pi, \nu \in \hat{K}$ be irreducible unitary representations. Let $(e_i^\pi)_{1 \leq i \leq \dim \pi}$ be an orthonormal basis of H_π and $(e_i^\nu)_{1 \leq i \leq \dim \nu}$ be an orthonormal basis of H_ν. Denote the associated matrix coefficients by $(e_{ij}^\pi)_{1 \leq i,j \leq \dim \pi}$ and $(e_{ij}^\nu)_{1 \leq i,j \leq \dim \nu}$. Theorem 8.1.11 implies that

$$\langle \chi_\pi, \chi_\nu \rangle = \langle \sum_i e_{ii}^\pi, \sum_j e_{jj}^\nu \rangle = \dim \pi \frac{1}{\dim \pi} \delta_{\pi,\nu} = \delta_{\pi,\nu}\,.$$

If π is a finite dimensional unitary representation of K and $\nu \in \hat{K}$, then we recover the multiplicity of ν in π by

$$\langle \chi_\pi, \chi_\nu \rangle = \sum_{\mu \in \hat{K}} m_\mu^\pi \langle \chi_\mu, \chi_\nu \rangle = m_\nu^\pi\,.$$

So the isomorphism class of π is determined by χ_π. ☐

Let us turn to two useful applications of characters.

Theorem 8.2.5. *Let K be a compact group. Then $(\chi_\pi)_{\pi \in \hat{K}}$ is an orthonormal basis for the space of conjugation invariant functions in $\mathrm{L}^2(K)$.*

Proof. We already observed in Remark 8.2.3 that characters are conjugation invariant. Let $\mathrm{L}^2{}_{\mathcal{Z}}(K) \subset \mathrm{L}^2(K)$ be the subspace of conjugation invariant elements. Denote by $p : \mathrm{L}^2(K) \to \mathrm{L}^2{}_{\mathcal{Z}}(K)$ the orthogonal projection. By Proposition 8.2.4 the characters of K form an orthonormal system in $\mathrm{L}^2{}_{\mathcal{Z}}(K)$. We have to show that they span a dense subspace of $\mathrm{L}^2{}_{\mathcal{Z}}(K)$. We are going to show that the orthonormal basis $(e_{ij}^\pi)_{\pi,i,j}$ of $\mathrm{L}^2(K)$ satisfies $pe_{ij}^\pi \in \mathbb{C}\chi_\pi$. Then the result follows thanks to the Peter-Weyl theorem 8.1.11. Since e_{ij}^π is a continuous function, the element pe_{ij}^π is represented by the function

$$(pe_{ij}^\pi)(x) = \int_K e_{ij}^\pi(k^{-1}xk)\mathrm{d}k\,.$$

We have

$$\begin{aligned}
\int_K e^\pi_{ij}(k^{-1}xk)\mathrm{d}k &= \int_K \langle \pi(x)\pi(k)e^\pi_i, \pi(k)e^\pi_j\rangle \mathrm{d}k \\
&= \sum_{l=1}^{\dim\pi} \int_K \langle \pi(x)\pi(k)e^\pi_i, e_l\rangle\langle e_l, \pi(k)e^\pi_j\rangle \mathrm{d}k \\
&= \sum_{l=1}^{\dim\pi} \int_K \langle \pi(k)e^\pi_i, \pi(x^{-1})e_l\rangle\langle e_l, \pi(k)e^\pi_j\rangle \mathrm{d}k \\
&= \sum_{l,r=1}^{\dim\pi} \int_K \langle \pi(k)e^\pi_i, e_r\rangle\langle e_r, \pi(x^{-1})e_l\rangle\langle e_l, \pi(k)e^\pi_j\rangle \mathrm{d}k \\
&= \sum_{l,r=1}^{\dim\pi} e^\pi_{lr}(x) \int_K e^\pi_{ir}(k)\overline{e^\pi_{jl}(k)}\mathrm{d}k \\
&= \frac{1}{\dim\pi}\sum_{l,r=1}^{\dim\pi} \delta_{ij}\delta_{rl}e^\pi_{lr}(x) \\
&= \frac{1}{\dim\pi}\delta_{ij}\sum_{l=1}^{\dim\pi} e^\pi_{ll}(x) \\
&= \frac{1}{\dim\pi}\delta_{ij}\chi_\pi(x)\,.
\end{aligned}$$

This finishes the proof of the theorem. □

Corollary 8.2.6. *Let F be a finite group. Then the number of conjugacy classes of F equals the number of isomorphism classes of irreducible representations of F.*

Proof. Theorem 8.2.5 says that the space of conjugation invariant functions in $\ell^2(F)$ has a basis of cardinality $|\hat{K}|$. Note that $\ell^2(F)$ is the space of all functions on F, because F is finite. Further, every conjugation invariant function on F is a unique linear combination of indicator functions on conjugacy classes in F. Thus, the dimension of $\ell^2(F)$ also equals the number of conjugacy classes in F. This proves the corollary. □

8.3 Operator algebras asssociated with compact groups

In this section we are going to describe the C*- and von Neumann algebras associated with compact groups. Let us start with the identification of the reduced and the maximal group C*-algebra.

Proposition 8.3.1. *Let K be a compact group. Then the canonical map $\lambda : \mathrm{C}^*_{\max}(K) \to \mathrm{C}^*_{\mathrm{red}}(K)$ is an isomorphism.*

Proof. Since λ is surjective, it suffices to show that it is isometric, which can be checked on elements from $\mathrm{L}^1(K) \subset \mathrm{C}^*_{\max}(K)$. If π is a unitary representation of G, then $\pi \leq (\dim\pi)\lambda$ by Peter-Weyl theory (e.g. Lemma 8.1.4). So for $f \in \mathrm{L}^1(K)$ we obtain

$$\|\pi(f)\| \leq \|\lambda^{\oplus\dim\pi}(f)\| = \|\lambda(f)\|\,,$$

since $\| \bigoplus_i T_i \| = \sup_i \|T_i\|$ for any family of bounded operators T_i acting on some Hilbert space H_i. We conclude that

$$\|f\|_{\mathrm{C}^*_{\max}} = \sup_{\pi} \|\pi(f)\| \leq \|\lambda(f)\| \leq \|f\|_{\mathrm{C}^*_{\max}} ,$$

where the supremum runs over all unitary representations π of K. This finishes the proof of the proposition. □

Remark 8.3.2. For reasons that will not be explained further, the fact that the natural *-homomophism $\mathrm{C}^*_{\max}(K) \to \mathrm{C}^*_{\mathrm{red}}(K)$ is an isomorphism is expressed by saying that "compact groups have the weak containment property". We write $\mathrm{C}^*(K)$ to denote this unique C*-algebra.

We are next going to decompose operator algebras associated with compact groups as direct sums of matrix algebra. A central role in this operation is played by characters.

Proposition 8.3.3. *Let K be a compact group and $\pi \in \hat{K}$. The function $(\dim \pi)\chi_\pi \in \mathrm{C}(K)$ is the projection onto the space spanned by matrix coefficients of π inside $\mathrm{L}^2(K)$. It is central in $\mathrm{C}(K)$.*

Proof. For $\pi, \nu \in \hat{K}$, let $(e_i^\pi)_{1 \leq i \leq \dim \pi}$ and $(e_i^\nu)_{1 \leq i \leq \dim \nu}$ be orthogonal bases of H_π and H_ν, respectively. Denote the associated matrix coefficients by $(e_{ij}^\pi)_{1 \leq i,j \leq \dim \pi}$ and $(e_{ij}^\nu)_{1 \leq i,j \leq \dim \nu}$. By Theorem 8.1.11 is suffices to check that $\lambda((\dim \pi)\chi_\pi)e_{ij}^\nu = \delta_{\pi,\nu} e^\pi$, for all $1 \leq i, j \leq \dim \nu$. We have

$$\lambda(\chi_\pi)e_{ij}^\nu = \chi_\pi * e_{ij}^\nu .$$

We further obtain

$$\begin{aligned}
\chi_\pi * e_{ij}^\nu(x) &= \int_K \chi_\pi(y) e_{ij}^\nu(y^{-1}x)\mathrm{d}y \\
&= \sum_{k=1}^{\dim \pi} \int_K e_{kk}^\pi(y) e_{ij}^\nu(y^{-1}x)\mathrm{d}y \\
&= \sum_{k=1}^{\dim \pi} \int_K e_{kk}^\pi(y) \langle \nu(x) e_i^\nu, \nu(y) e_j^\nu \rangle \mathrm{d}y \\
&= \sum_{k,r=1}^{\dim \pi} \int_K e_{kk}^\pi(y) \overline{e_{jr}^\nu(y)} \langle \nu(x) e_i^\nu, e_r^\nu \rangle \mathrm{d}y \\
&= \sum_{k,r=1}^{\dim \pi} \langle \nu(x) e_i^\nu, e_r^\nu \rangle \int_K e_{kk}^\pi(y) \overline{e_{jr}^\nu(y)} \mathrm{d}y \\
&= \frac{1}{\dim \pi} \delta_{\pi,\nu} \langle \pi(x) e_i^\nu, e_j^\nu \rangle \\
&= \frac{1}{\dim \pi} \delta_{\pi,\nu} e_{ij}^\pi(x) .
\end{aligned}$$

This shows that $(\dim \pi)\chi_\pi$ is a projection.

If $f \in \mathrm{C}(K)$ and $x \in K$, then Remark 8.2.3 implies that

$$\begin{aligned}(\chi_\pi * f)(x) &= \int_K \chi_\pi(y) f(y^{-1}x)\mathrm{d}y \\ &= \int_K \chi_\pi(xy) f(y^{-1})\mathrm{d}y \\ &= \int_K f(y)\chi_\pi(xy^{-1})\mathrm{d}y \\ &= \int_K f(y)\chi_\pi(y^{-1}x)\mathrm{d}y \\ &= (f * \chi_\pi)(x)\,.\end{aligned}$$

This shows that χ_π is central in $\mathrm{C}(K)$ and so is $(\dim \pi)\chi_\pi$. This finishes the proof of the proposition. □

Notation 8.3.4. Let K be a compact group and $\pi \in \hat{K}$. The projection $p_\pi = \sqrt{\dim \pi}\chi_\pi \in \mathrm{C}(K)$ is called the isotypical projection associated with π. It is an element of $\mathrm{C}^*(K)$ and of $\mathrm{L}(K)$.

We want to prove that isotypical projections give rise to natural approximate identities of group C^*-algebras in the sense of Definition 6.2.3. In order to do so, let's fix the following lemma.

Lemma 8.3.5. *Let K be a compact group and $f \in \mathrm{C}(K)$. Then*

$$\|\lambda(f)\| \leq \|f\|_1 \leq \|f\|_2\,.$$

Proof. The inequality $\|\lambda(f)\| \leq \|f\|_1$ was already proven in Proposition 6.3.3. Further, the Cauchy-Schwarz inequality implies that

$$\|f\|_1 = \|f\mathbb{1}_K\|_1 \leq \|f\|_2\|\mathbb{1}_K\|_2 = \|f\|_2\,.$$

This finishes the proof of the lemma. □

We next give a conceptual framework for the treatment of approximate identities in C^*-algebras. The next definition goes beyond the terminology of these notes, but it is immediately clarified by the following proposition.

Definition 8.3.6. Let A be a C^*-algebra. The multiplier algebra of A is the universal C^*-algebra $\mathrm{M}(A)$ that contains A as an essential ideal $A \trianglelefteq \mathrm{M}(A)$, that is if $mA = 0$, then $m = 0$ for all $x \in \mathrm{M}(A)$.

A net $(m_i) \in \mathrm{M}(A)$ converges to $m \in \mathrm{M}(A)$ strictly, if $m_i a \to ma$ and $am_i \to am$ in norm for all $a \in A$.

Proposition 8.3.7. *Let $A \subset \mathcal{B}(H)$ be a C^*-algebra with a non-degenerate representation. Then $A \subset \mathrm{M}(A)$ is isomorphic with $A \subset \{T \in \mathcal{B}(H) \mid \forall a \in A : Ta, aT \in A\}$.*

Remark 8.3.8. Let A be a C^*-algebra. It follows right from the definitions that a net $(a_i)_{i\in I}$ in A is an approximate identity of A if and only if $a_i \to 1$ strictly in $\mathrm{M}(A)$.

We have now developed the appropriate language to describe the isotypical projections as elements of the group C^*-algebra and of the group von Neumann algebra of a compact group.

Proposition 8.3.9. *Let K be a compact group. Then*

(i) $\sum_{\pi \in \hat{K}} p_\pi = 1$ *in the strict topology of* $\mathrm{M}(\mathrm{C}^*(K))$, *and*

(ii) $\sum_{\pi \in \hat{K}} p_\pi = 1$ *in the strong topology of* $\mathrm{L}(K)$.

Proof. We first show that $\sum_{\pi \in \hat{K}} p_\pi = 1$ in the strong topology of $\mathrm{L}(K)$. Since the strong topology of $\mathrm{L}(K)$ is the restriction of the strong topology of $\mathcal{B}(\mathrm{L}^2(K))$ is suffices appeal to Proposition 8.3.3 and the Peter-Weyl theorem 8.1.11.

We next prove that $\sum_{\pi \in \hat{K}} p_\pi = 1$ in the strict topology of $\mathrm{M}(\mathrm{C}^*(K))$. Since p_π are pairwise orthogonal projections by Proposition 8.3.3, the partial sums of $\sum_{\pi \in \hat{K}} p_\pi$ all have operator norm equal to one. Further, each summand is self-adjoint, so that $\left(\sum_{\pi \in F} p_\pi\right) a^* \to a^*$ implies that $a\left(\sum_{\pi \in F} p_\pi\right) \to a$ for all $a \in \mathrm{C}^*(K)$, where F ran through finite subsets of $\hat{K}$. We conclude that it suffices to check that $\sum_{\pi \in F} p_\pi \lambda(f) \to \lambda(f)$ for all $f \in \mathrm{C}(K)$.

Let $f \in \mathrm{C}(K)$ and $\varepsilon > 0$. By Proposition 8.3.3 and the Peter-Weyl theorem 8.1.11, there is some finite subset $F \subset \hat{K}$ such that $\| \sum_{\pi \in F} p_\pi f - f \|_2 < \varepsilon$. Then Lemma 8.3.5 implies that

$$\| \sum_{\pi \in F} p_\pi \lambda(f) - \lambda(f) \| = \| \lambda(\sum_{\pi \in F} p_\pi f - f \| \leq \| \sum_{\pi \in F} p_\pi f - f \|_2 < \varepsilon \,,$$

showing that $\sum_{\pi \in F} p_\pi \lambda(f) \to \lambda(f)$ indeed. So $\sum_{\pi \in \hat{K}} p_\pi = 1$ in the strict topology of $\mathrm{M}(\mathrm{C}^*(K))$. This finishes the proof of the proposition. ☐

We summarise the content of this section in the following concise statement describe the group C^*-algebra and the group von Neumann algebra of a compact group.

Notation 8.3.10. Let $(A_i)_{i \in I}$ be a family of C^*-algebras. Then

$$\mathrm{c}_0\text{-}\bigoplus_{i \in I} A_i = \{(a_i)_{i \in I} \in \Pi_{i \in I} A_i \mid \|a_i\| \to 0\}$$

is the C^*-algebraic direct sum of the $(A_i)_i$ and

$$\ell^\infty\text{-}\bigoplus_{i \in I} A_i = \{(a_i)_{i \in I} \in \Pi_{i \in I} A_i \mid \sup_{i \in I} \|a_i\| < \infty\} \,.$$

In case where all $(A_i)_i$ are von Neumann algebras, $\ell^\infty\text{-}\bigoplus_{i \in I} A_i$ is a von Neumann algebra too.

Theorem 8.3.11. *Let K be a compact group. Then*

$$\mathrm{C}^*(K) = \mathrm{c}_0\text{-}\bigoplus_{\pi \in \hat{K}} p_\pi \mathrm{C}^*(K) \cong \mathrm{c}_0\text{-}\bigoplus_{\pi \in \hat{K}} \mathrm{M}_{\dim \pi}(\mathbb{C})$$

and

$$\mathrm{L}(K) = \ell^\infty\text{-}\bigoplus_{\pi \in \hat{K}} p_\pi \mathrm{L}(K) \cong \ell^\infty\text{-}\bigoplus_{\pi \in \hat{K}} \mathrm{M}_{\dim \pi}(\mathbb{C}) \,.$$

Proof. We first treat $\mathrm{C}^*(K)$. Since $\sum_{\pi \in \hat{K}} p_\pi$ is an approximate unit in $\mathrm{C}^*(K)$ by Proposition 8.3.9 and the p_π, $\pi \in \hat{K}$ are pairwise orthogonal central projections in $\mathrm{C}^*(K)$, by Proposition 8.3.3, we obtain the equality

$$\mathrm{C}^*(K) = \mathrm{c}_0\text{-}\bigoplus_{\pi \in \hat{K}} p_\pi \mathrm{C}^*(K) \,.$$

We have to identify the direct summands $p_\pi \mathrm{C}^*(K)$. By Proposition 8.3.3 and the base-free version of the Peter-Weyl Theorem 8.1.14, we can naturally identify $p_\pi \mathrm{L}^2(K) \cong H_\pi \otimes \overline{H_\pi}$ and see that $p_\pi \mathrm{C}^*(K) \subset \mathcal{B}(H_\pi \otimes \overline{H_\pi}) = \mathcal{B}(H_\pi) \otimes \mathcal{B}(\overline{H_\pi})$ acting irreducibly on each subspace $H_\pi \otimes \overline{\eta}$, $\eta \in H_\pi$. So $p_\pi \mathrm{C}^*(K) \supset \mathcal{B}(H_\pi) \otimes 1$. A similar argument employing the right regular representation shows that $p_\pi \mathrm{C}^*(\rho(\mathrm{L}^1(K))) \supset 1 \otimes \mathcal{B}(\overline{H_\pi})$. Since $\mathrm{C}^*(K)$ and $\mathrm{C}^*(\rho(K))$ commute, this implies that $p_\pi \mathrm{C}^*(K) = \mathcal{B}(H_\pi) \otimes 1 \cong \mathrm{M}_{\dim \pi}(\mathbb{C})$. This finishes the C^*-algebraic part of the theorem.

As for the C^*-algebra, we see from Propositions 8.3.3 and 8.3.9 that $\mathrm{L}(K) = \ell^\infty\text{-}\bigoplus_{\pi \in \hat{K}} p_\pi \mathrm{L}(K)$. Further, $p_\pi \mathrm{L}(K) \subset P_\pi \mathrm{C}^*(K)$ is a dense and finite dimensional subalgebra. It follows that $p_\pi \mathrm{L}(K) = p_\pi \mathrm{C}^*(K) \cong \mathrm{M}_{\dim \pi}(\mathbb{C})$, which finishes the proof of the theorem. □

Remark 8.3.12. We appealed in the proof of Theorem 8.3.11 to the fact that the central pairwise orthogonal projections p_π, whose partial sums form an approximate identity of $\mathrm{C}^*(K)$ and of $\mathrm{L}(K)$ (in the strong topology) respectively give rise to a direct sum decomposition. This can be proven in a completely general C^*-algebraic and von Neumann algebraic setting, but will not be further investigated here.

9 Abelian groups: the Fourier transform and Pontryagin duality

In this section we are going to study unitary representations of abelian groups. It is partially based on Chapter 3 of [DE14]. It turns out that a generalisation of the classical Fourier transform (Section 9.3) and a certain duality theory termed Pontryagin duality (Section 9.5) are most suitable for this.

We recall that a character of a topological group G is a continuous group homomorphism $G \to \mathrm{S}^1$. Let us remark, that in the literature sometimes such characters are more precisely called unitary characters and a character is allowed to take values in all of the multiplicative group $\mathbb{C}^\times$. We denote by $\mathrm{Char}(G) = \{\chi : G \to \mathrm{S}^1 \mid \chi \text{ character}\}$ the set of all characters of G.

Let us start by the following observation, motivating the perspective taken in this section.

Proposition 9.0.1. *Let G be an abelian topological group. Then every irreducible representation of G is one-dimensional. In particular, there is a one-to-one correspondence between isomorphism classes of irreducible unitary representations and characters of G.*

Proof. Let $\pi : G \to \mathcal{U}(H)$ be an irreducible representation. By Proposition 3.2.4 (Schur's lemma), we have $\pi(G) \subset \pi(G)' \subset \mathbb{C}$. Let $\xi \in H$ be a non-zero vector. Since G is irreducible, we have $H = \overline{\mathrm{span}}\, \pi(G)\xi \subset \overline{\mathrm{span}}\, \mathrm{S}^1\xi = \mathbb{C}\xi$. This shows that H is one-dimensional.

Since $\mathrm{S}^1 \cong \mathcal{U}(H)$ canonically for every one-dimensional Hilbert space H, we obtain the claimed one-to-one correspondence between irreducible unitary representations and characters of G. □

In order to underline the important role of characters of abelian groups, let us mention the fact that every unitary representation of an abelian topological group admits a unique decomposition as a direct integral of irreducible unitary representations – a direct integral decomposition being a generalisation of a direct sum decomposition. In view of Proposition 9.0.1, this shows that we have to understand characters of abelian groups in order to understand its representation theory.

Before we proceed to the systematic study of characters of abelian locally compact groups, we argue that also characters of other groups are best studied in the setting of abelian groups.

Definition 9.0.2. Let G be a topological group. Then $[G,G] = \langle [g,h] \mid g,h \in G\rangle$ is called the (algebraic) commutator subgroup of G.

Proposition 9.0.3. *Let G be a topological group. Then commutator subgroup of G and its closure are normal subgroups of G.*

Proof. If $[G,G] \leq G$ is normal, so is its closure by Proposition 2.2.4. In order to prove this, it suffices to show that the set $\{[g,h] \mid g,h \in G\}$ is conjugation invariant. But this follows from the identity

$$\begin{aligned} x[g,h]x^{-1} &= xghg^{1}h^{-1}x^{1} = xg(x^{-1}x)h(x^{-1}x)g^{-1}(x^{-1}x)h^{-1}x^{-1} \\ &= (xgx^{-1})(xhx^{-1})(xgx^{-1})^{-1}(xhx^{-1})^{-1} = [xgx^{-1}, xhx^{-1}]. \end{aligned}$$ □

Definition 9.0.4. The abelianisation of a topological group G is $G^{\mathrm{ab}} = G/\overline{[G,G]}$.

Proposition 9.0.5. *Let G be a topological group and $\pi : G \to H$ a continuous homomorphism into an abelian topological group. Then π factors through $G \to G^{\mathrm{ab}}$ via a unique continuous homomorphism $G^{\mathrm{ab}} \to H$. In particular, every character of G factors through $G \to G^{\mathrm{ab}}$.*

Proof. Let $\pi : G \to H$ be a continuous homomorphism into an abelian topological group. Then $\ker \pi$ is closed in G. So it suffices to show that $[g,h] \in \ker \pi$ for all $g,h \in G$. This follows from the calculation

$$\pi([g,h]) = \pi(ghg^{-1}h^{-1}) = \pi(g)\pi(h)\pi(g)^{-1}\pi(h)^{-1} = [\pi(g),\pi(h)] = e_H,$$

which proves the proposition. □

9.1 Characters and the Pontryagin dual

In this section we are going to introduce the character group as a topological group. We are going to focus on the character groups of locally compact groups.

Proposition 9.1.1. *Let G be a topological group. Then* $\mathrm{Char}(G)$ *is an abelian group when equipped with pointwise multiplication and inversion. The constant function* $1_G \in \mathrm{Char}(G)$ *is the identity of* $\mathrm{Char}(G)$.

Proof. Since S^1 is a group, pointwise multiplication and inversion are well-defined operations on $\mathrm{Char}(G)$ and $\mathrm{Char}(G)$ becomes a group when equipped with these. Since S^1 is abelian, also $\mathrm{Char}(G)$ is abelian. Because $1 \in \mathrm{S}^1$ is the identity of S^1, it follows that 1_G is the identity of $\mathrm{Char}(G)$. □

We are next going to consider a natural topology on the character group.

Definition 9.1.2. Let X be locally compact space. The compact-open topology on $\mathrm{C}(X)$ is the coarsest topology containing the sets

$$\mathrm{L}(K,U) = \{f \in \mathrm{C}(X) \mid f(K) \subset U\}$$

for all $K \subset X$ compact and all $U \subset \mathbb{C}$ open.

Lemma 9.1.3. *Let X be a locally compact space. Then a net $(f_i)_{i\in I}$ in* $\mathrm{C}(X)$ *converges to a function $f \in \mathrm{C}(X)$ in the compact-open topology if and only if it converges uniformly on compact subsets of X.*

Proof. Let $(f_i)_{i\in I}$ be a net in $\mathrm{C}(X)$ and $f \in \mathrm{C}(X)$. We first assume that $f_i \to f$ in the compact-open topology. Let $K \subset X$ be compact and $\varepsilon > 0$. We have to show that there is $i_0 \in I$ such that for $i \geq i_0$ and $x \in K$ we have $|f_i(x) - f(x)| < \varepsilon$. Since $K \subset X$ is compact, X is locally compact and f is continuous, there are finitely many relatively compact open sets $U_1, \dots, U_n \subset X$ and there are $c_1, \dots, c_n \in \mathbb{C}$ such that $f(U_k) \subset \mathrm{B}_{\varepsilon/4}(c_j)$ for all $j \in \{1, \dots, n\}$ and such that $K \subset \bigcup_{j=1}^n U_k$. Let $K_j := \overline{U_j}$ be the compact closure of U_j for $j \in \{1, \dots, n\}$. Then $f(K_j) \subset \overline{\mathrm{B}_{\varepsilon/4}(c_j)} \subset \mathrm{B}_{\varepsilon/2}(c_j)$ for all $j \in \{1, \dots, n\}$, meaning that $\bigcap_{j=1}^n \mathrm{L}(K_j, \mathrm{B}_{\varepsilon/2}(c_j))$ is a neighbourhood of f in the compact-open topology. Take $i_0 \in I$ such that $f_i \in \bigcap_{j=1}^n \mathrm{L}(K_j, \mathrm{B}_{\varepsilon/2}(c_j))$ for all $i \geq i_0$. Then for $x \in K_j$ and $i \geq i_0$ we have $f_i(x)$, $f(x) \in \mathrm{B}_{\varepsilon/2}(c_j)$. So $|f_i(x) - f(x)| \leq \mathrm{diam}(\mathrm{B}_{\varepsilon/2}(c_j)) = \varepsilon$. We showed that $f_i \to f$ uniformly on compact subsets of X.

Assume now that $f_i \to f$ uniformly on compact subsets of X. Let $K \subset X$ be compact and $U \supset f(K)$ be open. Since f is continuous, $f(K) \subset \mathbb{C}$ is compact. So the lower semi-continuous function $c \mapsto \inf_{c' \in \mathbb{C} \setminus U} \mathrm{d}(c, c')$ attains a minimum on $f(K)$. It follows that there is $\varepsilon > 0$ such that $U \supset \bigcup_{c \in f(K)} \mathrm{B}_\varepsilon(c) \supset f(K)$. Take $i_0 \in I$ such that for $i \geq i_0$ and for $x \in K$ we have $|f_i(x) - f(x)| < \varepsilon$. Then $f(K) \subset U$. □

Proposition 9.1.4. *Let G be a topological group. Then* $\mathrm{Char}(G)$ *is a Hausdorff topological group with the compact-open topology.*

Proof. Let us first check that $\mathrm{Char}(G)$ is a topological group. We have to check that if $(\chi_{1,i})_{i\in I}$ and $(\chi_{2,i})_{i\in I}$ are convergent nets in $\mathrm{Char}(G)$ with limits χ_1 and χ_2, respectively, then $\chi_{1,i}\overline{\chi_{2,i}} \to \chi_1\overline{\chi_2}$. By Lemma 9.1.3, convergence in the compact-open topology is the same as uniform convergence on compact subsets of G, which implies the previous statement immediately. The fact that $\mathrm{Char}(G)$ is Hausdorff, follows away from the definition of the compact-open topology. □

Definition 9.1.5. Let G be a locally compact group. Then the topological group $\mathrm{Char}(G)$ is called the character group of G. If A is an abelian locally compact group, we also write $\hat{A} = \mathrm{Char}(A)$ and call it the Pontryagin dual of A.

It is important to recognise that the Pontryagin dual of a locally compact group is a locally compact group again. We employ operator algebraic "Gelfand duality" in order to see this.

Definition 9.1.6. Let A be a C^*-algebra. A character on A is a non-zero $*$-homomorphism $A \to \mathbb{C}$. If A is abelian, its spectrum $\hat{A}$, also denoted $\mathrm{spec}(A)$, is the set of characters of A equipped with the weak-$*$-topology inherited from the inclusion into the dual A^*.

The following theorem summarises a classical result in C^*-algebra theory.

Theorem 9.1.7 (Gelfand-Naimark duality). *For every abelian C^*-algebra A, the spectrum $\hat{A}$ is a locally compact Hausdorff space. The assignment $A \mapsto \hat{A}$ is functorial from the category of abelian C^*-algebras with non-degenerate $*$-homomorphisms to the category of locally compact Hausdorff spaces with proper continuous maps. Together with the functor $X \mapsto \mathrm{C}_0(X)$, it establishes an equivalence of categories.*

Let us recall some notions and results from earlier sections. First, if A is a locally compact abelian group, then Theorem 6.6.3 says that $\mathrm{C}^*_{\max}(A)$ is an abelian C^*-algebra. Further, every character of A can be identified with a one-dimensional unitary representation which in turn is identified with its integrated form on $\mathrm{C}^*_{\max}(A)$. So we obtain a bijection $\hat{A} \to \mathrm{spec}(\mathrm{C}^*_{\max}(A))$. Our aim is to show that this bijection is a homeomorphism. The next lemma establishes a correspondence between the weak-$*$-topology on $\mathrm{spec}(\mathrm{C}^*_{\max}(A))$ and the compact-open topology on $\hat{A}$.

Lemma 9.1.8. *Let A be a locally compact abelian group and χ_0 in $\hat{A}$. Let $C \subset A$ be a compact subset and let $\varepsilon > 0$. Then there are $f_0, f_1, \dots, f_n \in \mathrm{L}^1(A)$ and $\delta > 0$ such that for every $\chi \in \hat{A}$, the inequalities $|\chi(f_i) - \chi_0(f_i)| < \delta$ for all $i \in \{0, 1, \dots, n\}$ implies $|\chi(x) - \chi_0(x)| < \varepsilon$ for all $x \in C$.*

Proof. For all $\chi \in \hat{A}$ we have

$$|\chi(x) - \chi_0(x)| = |\overline{\chi_0}\chi(x) - 1|$$

and

$$|\chi(f_i) - \chi_0(f_i)| = |\overline{\chi_0}\chi(\chi_0 f_i) - 1_A(\chi_0 f)|$$

so that we can assume that $\chi_0 = 1_A$ is the trivial character. In what follows, we will use the equality

$$\chi(\mathrm{L}_x f) = \int_A \chi(y) f(xy) \mathrm{d}y = \int_A \chi(xy) f(y) \mathrm{d}y = \chi(x)\chi(f)$$

repeatedly.

Let $f \in \mathrm{L}^1(A)$ satisfy $1_A(f) = \int_A f(x)\mathrm{d}x = 1$. By Lemma 4.3.1 (iv) we can find a an identity neighbourhood $U \subset A$ such that $\|\mathrm{L}_x f - f\|_1 < \varepsilon/3$ for all $x \in U$. There are $x_1, \dots, x_n \in A$ such that $C \subset \bigcup_{i=1}^n x_i U$. Set $f_0 = f$ and $f_i = \mathrm{L}_{x_i} f$ for $i \in \{1, \dots, n\}$. Put $\delta = \varepsilon/3$. Let $\chi \in \hat{A}$ satisfy $|\chi(f_i) - 1| < \delta$ for all $i \in \{0, 1, \dots, n\}$. Let $x \in C$ and pick $i \in \{1, \dots, n\}$ such that $x \in x_i U$. Then we obtain the following estimates.

$$
\begin{aligned}
|\chi(x) - 1| &= |\overline{\chi}(x) - 1| \\
&\leq |\overline{\chi}(x) - \overline{\chi}(x)\chi(f)| + |\overline{\chi}(x)\chi(f) - \chi(f_i)| + |\chi(f_i) - 1| \\
&= |1 - \chi(f)| + |\chi(\mathrm{L}_x f) - \chi(\mathrm{L}_{x_i} f)| + |\chi(f_i) - 1| \\
&\leq \varepsilon/3 + \|\mathrm{L}_x f - \mathrm{L}_{x_i} f\|_1 + \varepsilon/3 \\
&\leq \varepsilon\,.
\end{aligned}
$$

□

Proposition 9.1.9. *Let A be a locally compact abelian group. The map associating with a character $\chi \in \hat{A}$ its integrated form on $\mathrm{C}^*_{\max}(A)$ is a homeomorphism $\hat{A} \to \mathrm{spec}(\mathrm{C}^*_{\max}(A))$.*

Proof. Let us first observe that there is indeed a bijection $\hat{A} \to \mathrm{spec}(\mathrm{C}^*_{\max}(A))$. By Proposition 9.0.1, characters of A are in bijection with unitary equivalence classes of one-dimensional unitary representations of A. Further, by Theorem 6.4.6 there is a bijection between unitary equivalence classes of one-dimensional unitary representations of A and non-zero *-homomorphisms $\mathrm{C}^*_{\max}(A) \to \mathbb{C}$. This establishes the aforementioned bijection $\hat{A} \to \mathrm{spec}\mathrm{C}^*_{\max}(A)$. It maps $\chi \in \hat{A}$ to the character satisfying $\chi(f) = \int_A \chi(x) f(x) \mathrm{d}x$ for all $f \in \mathrm{L}^1(A) \subset \mathrm{C}^*_{\max}(A)$.

We check that $\hat{A} \to \mathrm{spec}(\mathrm{C}^*_{\max}(A))$ is continuous and open. Assume that $(\chi_i)_{i \in I}$ is a net of characters in $\hat{A}$ converging to A in the compact-open topology. It suffices to show that $\chi_i(f) \to \chi(f)$ for all $f \in \mathrm{C}_c(A) \subset \mathrm{C}^*_{\max}(A)$. Take $f \in \mathrm{C}_c(A)$, let $\varepsilon > 0$ and let $i_0 \in I$ be chosen such that $|(\chi_i - \chi)(x)| < \varepsilon$ for all $x \in \mathrm{supp}\, f$. Then

$$
|(\chi_i - \chi)(f)| \leq \int_A |(\chi_i - \chi)(x)||f(x)| \mathrm{d}x < \varepsilon \|f\|_1\,.
$$

This shows that $\chi_i \to \chi$ in $\mathrm{spec}(\mathrm{C}^*_{\max}(A))$. Let us now show openness of $\hat{A} \to \mathrm{spec}(\mathrm{C}^*_{\max}(A))$. Let $(\chi_i)_{i \in I}$ be a net in $\hat{A}$ and $\chi \in A$ be such that $\chi_i \to \chi$ in $\mathrm{spec}(\mathrm{C}^*_{\max}(A))$. Let $C \subset A$ be compact and $\varepsilon > 0$. Let $\delta > 0$ and $f_0, \dots, f_n \in \mathrm{L}^1(A)$ be chosen according to Lemma 9.1.8 and let $i_0 \in I$ be chosen such that $|\chi_i(f_j) - \chi(f_j)| < \delta$ for all $j \in \{0, \dots, n\}$ and all $i \geq i_0$. Then the choice of δ and $f_0, \dots, f_n$ implies that $|\chi_i(x) - \chi(x)| < \varepsilon$ for all $x \in C$ and all $i \geq i_0$. This finishes the proof of the proposition. □

Let us now summarise the results of this section.

Theorem 9.1.10. *Let A be a locally compact abelian group. Then the Pontryagin dual $\hat{A}$ is a locally compact abelian group.*

Proof. By Proposition 9.1.4, we know that $\hat{A}$ is a Hausdorff topological group. Proposition 9.1.9 shows that $\hat{A}$ is homeomorphic with the spectrum of an abelian C*-algebra, which is locally compact by Theorem 9.1.7. This shows that $\hat{A}$ is indeed a locally compact abelian group. □

Example 9.1.11.

- The dual of $\mathbb{Z}$ is S^1, since every homomorphism $\mathbb{Z} \to \mathrm{S}^1$ is uniquely determined by the image of $1 \in \mathbb{Z}$.
- The dual of S^1 is $\mathbb{Z}$, since every continuous homomorphism $\varphi : \mathrm{S}^1 \to \mathrm{S}^1$ gives rise a multiplicative map $\mathbb{R} \to \mathrm{S}^1$, which can be derived at 0, showing that $\varphi(e^{it}) = e^{int}$ for some $n \in \mathbb{Z}$.
- The dual of $\mathbb{R}$ is $\mathbb{R}$, via the duality $\chi_t(s) = e^{ist}$.

Exercise 9.1.12. Show that the Pontryagin dual of a discrete abelian group is compact and that the Pontryagin dual of a compact abelian group is discrete.

9.2 The weak containment property for abelian groups

We mentioned the weak containment property in the context of compact groups in Remark 8.3.2. In this section we are going to prove the weak containment property for locally compact abelian groups A, meaning that the natural $*$-homomorphism $\mathrm{C}^*_{\max}(A) \to \mathrm{C}^*_{\mathrm{red}}(A)$ is an isomorphism.

Lemma 9.2.1. *Let G be a locally compact group, $\chi \in \mathrm{Char}(G)$ and $f \in \mathrm{L}^1(G)$. Then $\|f\|_{\mathrm{C}^*_{\mathrm{red}}(G)} = \|\chi f\|_{\mathrm{C}^*_{\mathrm{red}}(G)}$.*

Proof. Let $\xi, \eta \in \mathrm{L}^2(G)$. Then

$$\begin{aligned}\langle \lambda(\chi f)\xi, \eta\rangle &= \int_G \int_G \chi(y) f(y) \xi(y^{-1}x)\overline{\eta}(x) \mathrm{d}y \mathrm{d}x \\ &= \int_G \int_G f(y)\overline{\chi}(y^{-1}x)\xi(y^{-1}x)\chi(x)\overline{\eta}(x)\mathrm{d}y\mathrm{d}x \\ &= \langle \lambda(f)\overline{\chi}\xi, \overline{\chi}\eta\rangle .\end{aligned}$$

Since $\|\chi\xi\|_2 = \|\xi\|_2$ and $\|\chi\eta\|_2 = \|\eta\|_2$, this finishes the proof of the lemma. □

Theorem 9.2.2. *Let A be a locally compact abelian group. Then the natural map $\lambda : \mathrm{C}^*_{\max}(A) \to \mathrm{C}^*_{\mathrm{red}}(A)$ is an isomorphism.*

Proof. We first show that every character of $\mathrm{C}^*_{\max}(A)$ factors through $\mathrm{C}^*_{\mathrm{red}}(A)$. Let χ_0 be a character of $\mathrm{C}^*_{\mathrm{red}}(A)$ considered as a character of $\mathrm{C}^*_{\max}(A)$ after composition with λ. If $\chi \in \mathrm{spec}(\mathrm{C}^*_{\max}(A))$, then Lemma 9.2.1 shows

$$|\chi(f)| = |\chi_0(\overline{\chi_0}\chi f)| \le \|\overline{\chi_0}\chi f\|_{\mathrm{C}^*_{\mathrm{red}}(A)} = \|f\|_{\mathrm{C}^*_{\mathrm{red}}(A)} .$$

So χ indeed factors through $\mathrm{C}^*_{\mathrm{red}}(A)$.

Now assume that $x \in \mathrm{C}^*_{\max}(A)$ satisfies $\lambda(x) = 0$. And let $\chi \in \mathrm{spec}(\mathrm{C}^*_{\max}(A))$ be a character. Then χ factors through λ and hence $\chi(x) = 0$. Since $\mathrm{C}^*_{\max}(A)$ is abelian, characters separate its points, so that $x = 0$ follows. This shows that λ is injective and hence a $*$-isomorphism. □

Notation 9.2.3. In accordance with Remark 8.3.2, if A is an abelian locally compact group, then we write $\mathrm{C}^*(A) = \mathrm{C}^*_{\mathrm{red}}(A) \cong \mathrm{C}^*_{\max}(A)$ for the group C^*-algebra of A.

9.3 The Fourier transform

An important tool when studying abelian groups is the so called Fourier transform. Thanks to our operator algebraic setting, we can derive it from Gelfand-Naimark duality.

Definition 9.3.1. Let A be a locally compact abelian group. Let $^{-} : \mathrm{C}_0(\hat{A}) \to \mathrm{C}_0(\hat{A})$ be the isomorphism induced by conjugation of characters. The composition $\mathcal{F} : \mathrm{L}^1(A) \to \mathrm{C}^*(A) \to \mathrm{C}_0(\hat{A}) \xrightarrow{-} \mathrm{C}_0(\hat{A})$ is called Fourier transform. We write $\hat{f} = \mathcal{F}f$ for the Fourier transform of $f \in \mathrm{L}^1(A)$.

Let us fix for future use an integral formula for the Fourier transform of an L^1-function.

Proposition 9.3.2. *Let A be a locally compact abelian group. Then the Fourier transform satisfies*

$$\hat{f}(\chi) = \int_A f(x)\overline{\chi}(x)\mathrm{d}x ,$$

for all $f \in \mathrm{L}^1(A)$. In particular, $\chi \mapsto \int_A f(x)\overline{\chi}(x)\mathrm{d}x$ is a function in $\mathrm{C}_0(\hat{A})$.

Proof. Theorem 9.1.7 and the explanations following it show that for $\chi \in \hat{A}$, the isomorphism $\mathrm{C}^*(A) \to \mathrm{C}_0(\hat{A})$ carries the integrated form of χ to the evaluation homomorphism at χ. So $\hat{f}(\chi)$ is the integrated form of $\overline{\chi}$ evaluated at f. Hence

$$\hat{f}(\chi) = \int_A f(x)\overline{\chi}(x)\mathrm{d}x$$

as claimed by the proposition. □

Remark 9.3.3. Although implicit in the notation, it is important to note that the Fourier transform depends on the concrete choice of a Haar measure on an abelian locally compact group A through the definition of $\mathrm{L}^1(A)$.

9.4 The Plancherel measure

In order to understand the representation theory of a locally compact abelian group A, it suffices to understand its left-regular representation according to Theorem 9.2.2. Hence, our aim will be to decompose $\mathrm{L}^2(A)$ in terms of $\hat{A}$. It will turn out that after choice of a Haar measure on A there is a unique Haar measure on $\hat{A}$ such that for all $f \in (\mathrm{L}^1 \cap \mathrm{C}_0)(A)$ with $\hat{f} \in \mathrm{L}^1(\hat{A})$ we have

$$\int_{\hat{A}} \hat{f}(\chi)\mathrm{d}\chi = f(e)\,. \tag{9.1}$$

This will be the content of Theorem 9.5.11.

We start by defining a class of functions in $\mathrm{C}(A)$, which will be our main technical tool in this section.

Definition 9.4.1. Let A be a abelian locally compact group. Then

$$\mathrm{C}_0^*(A) = \overline{\iota((\mathrm{L}^1 \cap \mathrm{C}_0)(A))} \subset \mathrm{C}_0(A) \oplus_{\ell^\infty} \mathrm{C}_0(\hat{A})\,,$$

where $\mathrm{C}_0(A) \oplus_{\ell^\infty} \mathrm{C}_0(\hat{A})$ denotes the algebraic direct sum of $\mathrm{C}(A)$ and $\mathrm{C}_0(\hat{A})$ equipped with the norm $\|(f,\varphi)\|_{\ell^\infty} = \max\{\|f\|, \|\varphi\|\}$ and $\iota : (\mathrm{L}^1 \cap \mathrm{C}_0)(A) \to \mathrm{C}_0(A) \oplus_{\ell^\infty} \mathrm{C}_0(\hat{A})$ is defined by $\iota(f) = (f, \hat{f})$. We denote the norm of $\mathrm{C}_0^*(A)$ by $\|\cdot\|_0^*$.

Lemma 9.4.2. *Let A be an abelian locally compact group and denote by $\pi_0 : \mathrm{C}_0(A) \oplus_{\ell^\infty} \mathrm{C}_0(\hat{A}) \to \mathrm{C}_0(A)$ and $\pi_* : \mathrm{C}_0(A) \oplus_{\ell^\infty} \mathrm{C}_0(\hat{A}) \to \mathrm{C}_0(\hat{A})$ the coordinate projections. Then $\pi_0|_{\mathrm{C}_0^*(A)}$ and $\pi_*|_{\mathrm{C}_0^*(A)}$ are injective.*

Proof. Let $(f_0, f_*) \in \mathrm{C}_0^*(A)$ and interpret $f_* \in \mathrm{C}^*(A) \cong \mathrm{C}_0(\hat{A})$ via the Gelfand transform. We will show that $\langle f_*(\xi), \eta\rangle = \langle f_0 * \xi, \eta\rangle$ for all $\xi, \eta \in \mathrm{C}_c(A) \subset \mathrm{L}^2(A)$. Then $f_* = 0$ if and only if $f_0 = 0$ follows, proving the lemma.

Let $\xi, \eta \in \mathrm{C}_c(A)$. Let $(f_n)_n$ be a sequence in $(\mathrm{L}^1 \cap \mathrm{C}_0)(A)$ such that $f_n \to f_0$ uniformly and $f \to f_*$ in $\mathrm{C}^*(A)$. We have $f_n * \xi \to f_0 * \xi$ uniformly and $f_n * \xi = \lambda(f_n)\xi \to f_*(\xi)$ in $\mathrm{L}^2(A)$. Hence, using the fact that η has compact support, we obtain that $\langle f_n * \xi, \eta\rangle \to \langle f_0 * \xi, \eta\rangle$ and $\langle f_n * \xi, \eta\rangle \to \langle f_*(\xi), \eta\rangle$. This finishes the proof of the lemma. □

Notation 9.4.3. From now on, we will identity elements $f = (f_0, f_*) \in \mathrm{C}_0^*(A)$ with their first component, thereby considering $\mathrm{C}_0^*(A)$ as a subset of $\mathrm{C}_0(A)$. We write $\hat{f} = f_*$ for the second component.

We are next going to collect several results leading us to the proof of (9.1). Let us start by relating the evaluation at the identity with the some integral expression. It extends the statement of Lemma 8.1.1.

Lemma 9.4.4. *Let G be a unimodular locally compact group and $f, g \in \mathrm{L}^2(G)$. Then $f * g$ exists and defines an element in $(\mathrm{L}^2 \cap \mathrm{C}_0)(G)$. It satisfies $\|f * g\|_\infty \leq \|f\|_2 \|g\|_2$ and $\|f * g\|_2 \leq \|f\|_2 \|g\|_2$ and $(f * f^*)(e) = \|f\|_2^2$.*

Proof. Since A is unimodular, Lemma 8.1.1 says that the convolution $f * g$ is well-defined and continuous. Recall that $(f * g)(x) = \langle f, \mathrm{L}_x g^* \rangle$ for every $x \in G$. The Cauchy-Schwarz inequality and unimodularity show that

$$\|f * g\|_\infty = \sup_{x \in G} |\langle f, \mathrm{L}_x g^* \rangle| \leq \sup_{x \in G} \|f\|_2 \|\mathrm{L}_x g^*\|_2 = \|f\|_2 \|g\|_2 \,.$$

If $(f_n)_n$ and $(g_n)_n$ are sequences in $\mathrm{C}_\mathrm{c}(G)$ converging in $\mathrm{L}^2(G)$ to f and g, respectively, then $f_n * g_n \in \mathrm{C}_\mathrm{c}(G)$ converges uniformly to $f * g$, showing that $f * g \in \mathrm{C}_0(A)$.

Next, we obtain that

$$\begin{aligned}
\|f * g\|_2^2 &= \int_G |f * g(x)|^2 \mathrm{d}x \\
&= \int_G |\int_G f(y) g(y^{-1}x) \mathrm{d}y|^2 \mathrm{d}x \\
&\leq \int_G \int_G |f(y) g(y^{-1}x)|^2 \mathrm{d}y \mathrm{d}x \\
&= \int_G \int_G |f(y) g(x)|^2 \mathrm{d}x \mathrm{d}y \\
&= \|f\|_2^2 \|g\|^2 \,.
\end{aligned}$$

We finally obtain that

$$(f * f^*)(e) = \int_A f(x) f^*(x^{-1}) \mathrm{d}x = \int_A |f(x)|^2 \mathrm{d}x = \|f\|_2^2 \,,$$

finishing the proof of the lemma. □

The next lemma controls, in which function spaces certain convolution products lie.

Lemma 9.4.5. *Let A be a locally compact abelian group.*

- $\mathrm{L}^1(A) * \mathrm{C}_\mathrm{c}(A) \subset \mathrm{C}_0(A)$
- $\lambda(\mathrm{C}^* A)(\mathrm{C}_\mathrm{c}(A) * \mathrm{C}_\mathrm{c}(A)) \subset \mathrm{L}^2(A) \cap \mathrm{C}_0^*(A)$.
  Further, if $f \in \mathrm{C}^*(A)$ and $g, h \in \mathrm{C}_\mathrm{c}(A)$, then $\mathcal{F}(\lambda(f)(g * h)) = \hat{f} \hat{g} \hat{h}$.

Proof. Let us prove that $f * g \in \mathrm{C}_0(A)$ if $f \in \mathrm{L}^1(A)$ and $g \in \mathrm{C}_\mathrm{c}(A)$. To this end let $(f_n)_n$ be a sequence in $\mathrm{C}_\mathrm{c}(A)$ converging to f in $\mathrm{L}^1(A)$. Then $f_n * g \in \mathrm{C}_\mathrm{c}(A)$ and

$$|(f * g - f_n * g)(x)| = |(f - f_n) * g(x)| \leq \|f - f_n\|_1 \|g\|_\infty \to 0 \,.$$

So $f_n * g \to f * g$ uniformly. This shows that $\mathrm{L}^1(A) * \mathrm{C_c}(A) \subset \mathrm{C_c}(A)$.

Let now $f \in \mathrm{C}^*(A)$ and $g, h \in \mathrm{C_c}(A)$. If is clear that $\lambda(f)(g * h) \in \mathrm{L}^2(A)$. Let $(f_n)_n$ be a sequence in $\mathrm{L}^1(A)$ approximating f in the C^*-norm. Then $\lambda(f_n)(g * h) = f_n * g * h \in (\mathrm{L}^1 \cap \mathrm{C}_0)(A)$ for all $n \in \mathbb{N}$. We show that $(f_n * g * h)_n$ converges to $\lambda(f)(g * h)$ in $\mathrm{C}_0(A)$ and $\mathcal{F}(f_n * g * h)_n$ converges to $\hat{f}\hat{g}\hat{h}$ in $\mathrm{C}_0(\hat{A})$. Uniform convergence in $\mathrm{C}_0(A)$ follows from Lemma 9.4.4 via the estimate

$$\begin{aligned}\|\lambda(f)(g * h) - \lambda(f_n)(g * h)\|_\infty &= \|\lambda(f - f_n)(g * h)\|_\infty \\ &\leq \|\lambda(f - f_n)g\|_2 \|h\|_2 \\ &\leq \|\lambda(f - f_n)\|_{\mathrm{C}^*(A)} \|g\|_2 \|h\|_2 \\ &\to 0 .\end{aligned}$$

Further, the fact that $\hat{f}_n$ converges uniformly to $\hat{f}$ and the identity $\mathcal{F}(f_n * g * h) = \hat{f}_n\hat{g}\hat{h}$ imply that $\mathcal{F}(\lambda(f)(h * g)) = \hat{f}\hat{g}\hat{h}$. This finishes the proof of the lemma. ☐

The next lemmas describes further instances in which a Dirac net provides an approximate identity of several operator algebras. The first lemma can be considered as a generalisation of Lemma 6.2.4, which treats the case of $\mathrm{L}^1(G)$ for a locally compact group G.

Lemma 9.4.6. *Let G be a locally compact group and $(f_U)_U$ a Dirac net for G. Then $(f_U)_U$ is an approximate identity for $\mathrm{C}^*(G)$ and for the convolution action on $\mathrm{C_c}(G)$.*

Proof. By Lemma 6.2.4, we have an approximate identity $(f_U)_U$ of $\mathrm{L}^1(G)$. Since $\mathrm{L}^1(G) \subset \mathrm{C}^*(G)$ is dense and $\|f_U\|_{\mathrm{C}^*(G)} \leq \|f_U\|_1 \leq 1$ for all U, we find that $(f_U)_U$ is an approximate identity for $\mathrm{C}^*(G)$.

Since convolution with f_U acts as a contraction on $\mathrm{C}_0(G)$ and $(f_U * g)^* = g^* * f_U^*$ it suffices to prove that $f_U * g \to g$ uniformly for all $g \in \mathrm{C_c}(A)$. If $x \in G$ and $U \subset G$ is an identity neighbourhood such that $|g(y^{-1}x) - g(x)|$ for all $y \in U$, then

$$(f_U * g - g)(x) = |\int_A f_U(y)g(y^{-1}x) - g(x)\mathrm{d}x| \leq \int_A |f_U(y)(g(y^{-1}x) - g(x))|\mathrm{d}x \leq \varepsilon \int_G f_U(y)\mathrm{d}y = \varepsilon .$$

Since g is uniformly continuous by Lemma 4.1.2, this shows $f_u * g \to g$ uniformly. ☐

Lemma 9.4.7. *Let A be a locally compact abelian group and let $(f_U)_U$ be a Dirac net in $\mathrm{C_c}(A)$. Then $(f_U)_U$ acting by convolution on $\mathrm{C}_0^*(A)$ is an approximate identity. Further, $(\hat{f}_U)_U$ converges uniformly on compact subsets to the constant function 1 on $\hat{A}$.*

Proof. Let $g \in \mathrm{C}_0^*(A)$. Then Lemma 9.4.6 says that $f_U * g \to g$ in the norm of $\mathrm{C}_0(A)$. Further, $\widehat{f_U * g} = \hat{f}_U\hat{g} \to \hat{g}$ in the norm of $\mathrm{C}_0(\hat{A}) \cong \mathrm{C}^*(A)$ by Lemma 9.4.6. This shows that $(f_U)_U$ is an approximate identity for $\mathrm{C}_0^*(A)$.

Let $C \subset \hat{A}$ be a compact subset and let $g \in \mathrm{C_c}(\hat{A})$ be a some function such that $g|_C \equiv 1$. Then $\hat{f}_U g \to g$ in $\mathrm{C}_0(\hat{A})$ by Lemma 9.4.6, showing that $\hat{f}_U \to 1$ uniformly on C. ☐

We arrive now at our main approximation result.

Lemma 9.4.8. *Let A be a locally compact abelian groups. Let $\varphi \in \mathrm{C_c}(\hat{A})$ be real valued and $\varepsilon > 0$. Then there are $f_1, f_2 \in (\mathrm{L}^2 \cap \mathrm{C}_0^*)(A)$ such that*

- *$\hat{f}_1, \hat{f}_2$ are real valued functions with support contained in $\mathrm{supp}(\varphi)$,*
- *$\hat{f}_1 \leq \varphi \leq \hat{f}_2$ and $\|\hat{f}_2 - \hat{f}_1\|_{\hat{A}} < \varepsilon$, and*

- $0 \leq f_2(1) - f_1(1) < \varepsilon$.

Proof. Put $K = \mathrm{supp}(\varphi)$ and let $1 > \delta > 0$. By Lemma 9.4.7, there is a function $g_\delta \in \mathrm{C_c}(A)$ such that $\sqrt{1-\delta} \leq |\hat{g}_\delta| \leq \sqrt{1+\delta}$ on K. Then $h_\delta = g_\delta * g_\delta^*$ satisfies $\hat{h}_\delta = |g_\delta|^2$ and hence $1 - \delta \leq \hat{h}_\delta|_K \leq 1 + \delta$. We can also fix some positive multiple of a Dirac function $g \in \mathrm{C}(A)$ such that $h = g * g^*$ satisfies $h|_K \geq 1$. Take $f \in \mathrm{C}^*(A)$ such that $\hat{f} = \varphi$ and put

$$f_1 = \lambda(f)(h_\delta - \delta h) \qquad f_2 = \lambda(f)(h_\delta + \delta h)\,.$$

By Lemma 9.4.5, we have $f_1, f_2 \in (\mathrm{L}^2 \cap \mathrm{C}_0)(A)$. Further, this lemma says that

$$\hat{f}_1(\chi) = \hat{f}(\chi)(\hat{h}_\delta(\chi) - \delta\hat{h}(\chi)) \leq \varphi(\chi) \leq \hat{f}_2(\chi)\,.$$

Further, $\mathrm{supp}(\hat{f}_i) \subset \mathrm{supp}(\varphi)$ for $i \in \{1, 2\}$ again using Lemma 9.4.5. The approximation properties of the lemma's statement follow by choosing δ small enough. □

We show that some positivity that would be implied by (9.1) can indeed be proved.

Lemma 9.4.9. *Let A be a locally compact abelian group and $f \in \mathrm{C}_0^*(A)$. If $\hat{f}$ is real-valued, then $f(e)$ is real. If $\hat{f} \geq 0$, then $f(e) \geq 0$.*

Proof. First assume that $\hat{f}$ is real-valued. Then $\hat{f} = \overline{\hat{f}} = \widehat{f^*}$, showing that $f = f^*$. So $f(e) = f^*(e) = \overline{f}(e)$.

Now assume that $\hat{f} \geq 0$. The composition with $\overline{} : \mathrm{C}_0(\hat{A}) \to \mathrm{C}_0(\hat{A})$ and the identification $\mathrm{C}_0(A) \cong \mathrm{C}^*(A)$ provides us with a non-negative element $f' \in \mathrm{C}^*(A)$ such that $f * h = \lambda(f')h$ for all $h \in \mathrm{C_c}(A)$. Let $g \in \mathrm{C}^*(A)$ be non-negative such that $g^2 = f'$. We find a sequence $(g_n)_n$ of self-adjoint elements in $\mathrm{L}^1(A)$ such that g_n converges to g in the C^*-norm. For any $h \in \mathrm{C_c}(A)$ we have $\lambda(g_n)h \to \lambda(g)h$ in $\mathrm{L}^2(A)$ and hence $(\lambda(g_n)h) * (\lambda(g_n)h)^* \to \lambda(g)h * (\lambda(g)h)^*$ uniformly by Lemma 9.4.4. Using commutativity of $\mathrm{C}^*(A)$, which is the content of Theorem 6.6.3, we obtain in the topology of uniform convergence

$$\begin{aligned}
\lambda(g)h * (\lambda(g)h)^* &= \lim \lambda(g_n)h * (\lambda(g_n)h)^* \\
&= \lim g_n * h * g_n^* * h^* \\
&= \lim g_n * g_n * h * h \\
&= \lim \lambda(g_n * g_n)(h * h) \\
&= \lambda(g^2)(h * h) \\
&= \lambda(f')(h * h) \\
&= f * h * h\,.
\end{aligned}$$

By Lemma 9.4.4 this implies that

$$(f * h * h)(e) = (\lambda(g)h * (\lambda(g)h)^*)(e) = \|\lambda(g)h\|_2^2 \geq 0\,.$$

Letting $h * h$ run through a Dirac net in $\mathrm{C_c}(A)$, Lemma 9.4.6 implies that $f(e) \geq 0$. □

We are now ready to define prove the formula (9.1) defines a Haar measure on $\hat{A}$ for a locally compact abelian group A.

Lemma 9.4.10. *Let A be a locally compact abelian group and $\varphi \in \mathrm{C_c}(\hat{A})$ be a real-valued function. Then*

$$\sup\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \leq \varphi\} = \inf\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \geq \varphi\}.$$

Proof. Lemma 9.4.9 says that if $f, g \in \mathrm{C_c}(\hat{A})$ satisfy $\hat{f} \leq \varphi \leq \hat{g}$, then $\hat{g} - \hat{f} = \widehat{g - f} \geq 0$. So

$$\sup\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \leq \varphi\} \leq \inf\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \geq \varphi\}.$$

Lemma 9.4.8 then shows equality. □

Proposition 9.4.11. *Let A be a locally compact abelian group. Then $I : \mathrm{C_c}(\hat{A}) \to \mathbb{C}$ defined by*

$$I(\varphi) = \sup\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \leq \operatorname{Re}\varphi\} + i \sup\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \leq \operatorname{Im}\varphi\}$$

is a Haar integral on $\hat{A}$.

Proof. Linearity of I follows from its definition. Further, I is positive, since $0_{\hat{A}} \leq \varphi$ for all positive functions $\varphi \in \mathrm{C_c}(\hat{A})$ and $\hat{0}_A = 0_{\hat{A}}$. Also, I is non-zero by Lemma 9.4.10. So it suffices to prove invariance of the expression $\sup\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \leq \varphi\}$ for real valued $\varphi \in \mathrm{C_c}(\hat{A})$. To this end let $\chi \in \hat{A}$. Then $\mathrm{L}_\chi \hat{f} = \widehat{\chi f}$ for all $f \in \mathrm{C_c}(A)$. This shows that

$$\begin{aligned}\sup\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \leq \mathrm{L}_\chi\varphi\} &= \sup\{f(e) \mid f \in \mathrm{C}_0^*(A), \mathrm{L}_{\overline{\chi}}\hat{f} \leq \varphi\} \\ &= \sup\{(\overline{\chi}f)(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \leq \varphi\} \\ &= \sup\{f(e) \mid f \in \mathrm{C}_0^*(A), \hat{f} \leq \varphi\}.\end{aligned}$$

This finishes the proof of the proposition. □

Remark 9.4.12. The Haar integral on $\hat{A}$ provided by Proposition 9.4.11 depends on the choice of a Haar measure on A. If μ is a fixed Haar measure on A and $c > 0$, then for a real valued function $\varphi \in \mathrm{C_c}(\hat{A})$ we have

$$\begin{aligned}I_\mu(\varphi) &= \sup\{f(1) \quad \mid f \in \mathrm{C}_0^*(A), \mathcal{F}_\mu(f) \leq \operatorname{Re}\varphi\} \\ &= \sup\{cf(1) \mid f \in \mathrm{C}_0^*(A), \mathcal{F}_\mu(cf) \leq \operatorname{Re}\varphi\} \\ &= c\sup\{f(1) \mid f \in \mathrm{C}_0^*(A), \mathcal{F}_c\mu(f) \leq \operatorname{Re}\varphi\} \\ &= cI_{c\mu}(\varphi).\end{aligned}$$

So $I_{c\mu} = \frac{1}{c} I_\mu$.

Definition 9.4.13. Let A be a locally compact abelian group with a fixed Haar measure μ. Then the Haar measure on $\hat{A}$ constructed in Proposition 9.4.11 is called the Plancherel measure on $\hat{A}$ associated with μ.

Notation 9.4.14. From now on integration on the Pontryagin dual of a locally compact abelian group A will always be with respect to the Plancherel measure associated with a fixed Haar measure on A.

Remark 9.4.15. Let A be a locally compact abelian group. By construction, the Plancherel measure on $\hat{A}$ satisfies

$$\int_{\hat{A}} \hat{f}(\chi) \mathrm{d}\chi = f(e)$$

for all $f \in \mathrm{C}_0^*(A)$ such that $\hat{f} \in \mathrm{C_c}(\hat{A})$. We will see in Theorem 9.5.11, that the second condition can be relaxed to $\hat{f} \in \mathrm{L}^1(\hat{A})$, which is optimal.

9.5 The Pontryagin dual

The aim of this section to prove a natural isomorphism of A with its double dual $\hat{\hat{A}}$ for all locally compact abelian groups A. Let us start by providing the natural map which provides this isomorphism.

Lemma 9.5.1. *Let A be a locally compact abelian group. For every $x \in A$, the map $\delta_x : \hat{A} \to S^1$ defined by $\delta_x(\chi) = \chi(x)$ is a continuous character of $\hat{A}$. The map $\delta : x \mapsto \delta_x$ is a continuous group homomorphism from A to $\hat{\hat{A}}$.*

Proof. Let us first show that for every $x \in A$ the map δ_x is a continuous character of $\hat{A}$. To this end let $\chi_1, \chi_2 \in \hat{A}$. Then

$$\begin{aligned}\delta_x(\chi_1\overline{\chi_2}) &= (\chi_1\overline{\chi_2})(x)\\ &= \chi_1(x)\overline{\chi_2(x)}\\ &= \delta_x(\chi_1)\overline{\delta_x(\chi_2)}\,.\end{aligned}$$

This shows that δ_x is a character of $\hat{A}$. Let now $(\chi_i)_i$ be a net in $\hat{A}$ converging to $\chi \in \hat{A}$. Then uniform convergence on compact subsets of A (Lemma 9.1.3) implies in particular $\chi_i(x) \to \chi(x)$. So $\delta_x(\chi_i) \to \delta_x(\chi)$. This proves that δ_x is continuous. Up to now we showed that δ is a well-defined map $A \to \hat{\hat{A}}$.

Let us show that δ is a continuous group homomorphism. To this end let $x, y \in A$ and $\chi \in \hat{A}$. Then

$$\begin{aligned}(\delta_{x\overline{y}})(\chi) &= \chi(x\overline{y})\\ &= \chi(x)\overline{\chi(y)}\\ &= \delta_x(\chi)\overline{\delta_y(\chi)}\\ &= \delta_x(\chi)\overline{\delta_y}(\chi)\,.\end{aligned}$$

This shows that δ is a group homomorphism. In order to show its continuity, we let $(x_i)_i$ be a net in A converging to $x \in A$. Let $C \subset \hat{A}$ be a compact subset. We have $\delta_{x_i}(\chi) \to \delta_x(\chi)$ for all $\chi \in C$ and since C is compact and all δ_{x_i} are continuous, it follows that $\delta_{x_i} \to \delta_x$ uniformly on C. This shows $\delta_{x_i} \to \delta_x$ in $\hat{\hat{A}}$ thanks to Lemma 9.1.3. □

Definition 9.5.2. Let A be a locally compact abelian group. The map $\delta : A \to \hat{\hat{A}}$ defined by $\delta_x(\chi) = \chi(x)$ for all $x \in A$ and all $\chi \in \hat{A}$ is called the Pontryagin map.

The Pontryagin map allows us to express the relation between Fourier transform and left-translation action in a convenient way.

Lemma 9.5.3. *Let A be a locally compact abelian group $x \in A$ and $f \in C^*(A)$. Then $\mathcal{F}(L_x f) = \overline{\delta_x}\hat{f}$.*

Proof. Let us first assume that $f \in L^1(A)$. Then

$$\mathcal{F}(L_x f)(\chi) = \int_A f(x^{-1}y)\overline{\chi(y)}\mathrm{d}y = \int_A f(y)\overline{\chi(xy)}\mathrm{d}y = (\overline{\delta_x}\hat{f})(\chi)\,,$$

for all $\chi \in \hat{A}$ shows that $\mathcal{F}(L_x f) = \overline{\delta_x}\hat{f}$ indeed. The lemma follows now by density of $L^1(A) \subset C^*(A)$ and continuity of the Fourier transform. □

We can right away see that the Pontryagin map is injective. This is the content of the following proposition.

Lemma 9.5.4. *Let A be a locally compact abelian group. Then the Pontryagin map of A is injective.*

Proof. Let $x \in A$ such that $\delta_x = 1_{\hat{A}}$. Then $\mathcal{F}(\mathrm{L}_x g) = \overline{\delta_x}\hat{g} = \hat{g}$ by Lemma 9.5.3 and hence $\mathrm{L}_x g = g$ for all $g \in \mathrm{L}^1(A)$. This shows $x = e$. □

In order to prove that the Pontryagin map is surjective, we are going to prove that it is closed and has a dense range. The next lemma provides us with a useful criterion for closedness.

Lemma 9.5.5. *Let $\varphi : X \to Y$ be a continuous map between locally compact spaces. Assume that φ is proper, that is $\varphi^{-1}(K)$ is compact for every compact subset $K \subset Y$. Then φ is closed.*

Proof. Let $(y_i)_i$ be a net in $\operatorname{im}\varphi$ converging to $y \in Y$. Let x_i be a preimage for y_i for every i. Since Y is locally compact, there is a compact neighbourhood K of y and we may assume that $y_i \in K$ for all i. Since $\varphi^{-1}(K)$ is compact, we may replace $(x_i)_i$ by a subnet and assume that $x_i \to x \in K$ converges. Then $y = \lim_i y = \lim_i \varphi(x_i) = \varphi(x)$ follows from continuity of φ and proves that $y \in \operatorname{im}\varphi$. This finishes the proof of the lemma. □

Let us fix a an inversion formula for the Pontryagin map, which holds à priori for a restricted class of functions.

Lemma 9.5.6. *Let A be a locally compact abelian group and $f \in \mathrm{C}_0^*(A)$ such that $\hat{f} \in \mathrm{C}_\mathrm{c}(\hat{A})$. Then $f(x) = \hat{\hat{f}}(\delta_{x^{-1}})$ for every $x \in A$.*

Proof. Thanks to Theorem 9.5.11 and Lemma 9.5.3, we have

$$f(x) = (\mathrm{L}_{x^{-1}} f)(e) = \int_{\hat{A}} \mathcal{F}(\mathrm{L}_{x^{-1}} f)(\chi)\mathrm{d}\chi = \int_{\hat{A}} \overline{\delta_{x^{-1}}}(\chi)\hat{f}(\chi)\mathrm{d}\chi = \hat{\hat{f}}(\delta_{x^{-1}})\,.$$

□

The next lemma will allow us to make use of the inversion formula by means of approximation.

Lemma 9.5.7. *Let A be a locally compact abelian group. Then both $\mathrm{C}_\mathrm{c}(A) \subset \mathrm{C}_0^*(A)$ is dense and $\mathrm{C}_\mathrm{c}(\hat{A}) \cap \{\hat{f} \mid f \in (\mathrm{L}^2 \cap \mathrm{C}_0^*)(A)\} \subset \mathrm{C}_0^*(\hat{A})$ is dense.*

Proof. Since $(\mathrm{L}^1 \cap \mathrm{C}_0)(A)$ is dense in $\mathrm{C}_0^*(A)$, it suffices to show that $\mathrm{C}_\mathrm{c}(A)$ is dense in the former for the norm $\|\cdot\|_0^*$. Let $f \in (\mathrm{L}^1 \cap \mathrm{C}_0)(A)$. For $n \in \mathbb{N}$ let $K_n \subset A$ be a compact subset such that $|f|_{A \setminus K_n}| \leq 1/n$. Let $g_n \in \mathrm{C}_\mathrm{c}(A)$ be functions such that $0 \leq g_n \leq 1$ and $g_n|_{K_n} \equiv 1$. Then $g_n f \to f$ uniformly and $g_n f \to f$ in $\|\cdot\|_1$. So also $g_n f \to f$ in $\|\cdot\|_{\mathrm{C}^*(A)}$. This shows that $\mathrm{C}_\mathrm{c}(A) \subset \mathrm{C}_0^*(A)$ is dense.

Now it also follows that $\mathrm{C}_\mathrm{c}(\hat{A}) \cap \{\hat{f} \mid f \in (\mathrm{L}^2 \cap \mathrm{C}_0^*)(A)\}$ is dense, since $\mathrm{C}_\mathrm{c}(\hat{A}) \subset \mathrm{C}_0^*(\hat{A})$ is dense by the first part of the lemma, and $\mathrm{C}_\mathrm{c}(\hat{A}) \cap \{\hat{f} \mid f \in (\mathrm{L}^2 \cap \mathrm{C}_0^*)(A)\} \subset \mathrm{C}_\mathrm{c}(\hat{A})$ is dense in the $\|\cdot\|_0^*$ norm by Lemma 9.4.8. This finishes the proof of the lemma. □

Theorem 9.5.8 (Pontryagin duality). *Let A be a locally compact abelian group. Then the Pontryagin map $\delta : A \to \hat{\hat{A}}$ is an isomorphism of topological groups.*

Proof. By Lemmas 9.5.1 and 9.5.4, we know already that δ is injective and continuous group homomorphism. So it remains to show that it is closed and surjective.

Let us start by showing that δ is proper, which together with Lemma 9.5.5 will show closedness of δ. Let $K \subset \hat{\hat{A}}$ be a compact subset. Applying Lemma 9.4.8 to some positive function in $\mathrm{C_c}(\hat{\hat{A}})$ that is constantly equal to 1 on K, we obtain some $\varphi \in \mathrm{C}_0^*(\hat{A})$ such that $\hat{\varphi}$ is positive and $\hat{\varphi}|_K \geq 1$. By Lemma 9.5.7 there is some element $\psi \in \mathrm{C_c}(\hat{A})$ such that $\|\varphi - \psi\|_0^* < 1/2$ and ψ is the Fourier transform of some element $f \in \mathrm{C}_0^*(A)$. We fix the compact set $C = \{x \in A \mid |f(x)| > 1/2\}$. If $\delta_x \in K$ for some $x \in A$, then Lemma 9.5.6 implies that

$$|f(x^{-1})| = |\hat{\hat{f}}(\delta_x)| = |\hat{\psi}(\delta_x)| > \hat{\varphi}(\delta_x) - \frac{1}{2} \geq \frac{1}{2}.$$

This shows that $x \in C^{-1}$ and thus δ is proper.

Let us next show that the image of δ is dense. For a contradiction, we assume that there is some non-empty open subset $U \subset \hat{\hat{A}}$ such that $U \cap \operatorname{im} \delta = \emptyset$. By Lemma 9.4.8 applied to any function in $\mathrm{C_c}(\hat{\hat{A}})$ whose support lies in U, we obtain a non-zero element $\varphi \in \mathrm{C}_0^*(\hat{A})$ such that $\hat{\varphi}$ is supported in U. By Lemma 9.5.7 there is a sequence $(\varphi_n)_n$ in $\mathrm{C_c}(\hat{A})$ such that $\varphi_n \to \varphi$ and such that $\varphi_n = \hat{f}_n$ for some $f_n \in \mathrm{C}_0^*(A)$. Lemma 9.5.6 applies together with the fact that images of compact sets under δ are compact to show that $f_n(x) = \hat{\hat{f}}_n(\delta_{x^{-1}}) = \hat{\varphi}_n(\delta_{x^{-1}}) \to \hat{\varphi}(\delta_{x^{-1}}) = 0$ uniformly. So $f_n \to 0$ in $\mathrm{C}_0(A)$. Further, $\hat{f}_n = \varphi_n \to \varphi$ in $\mathrm{C}_0(\hat{A})$, meaning that $(f_n)_n$ is a Cauchy sequence in $\mathrm{C}_0^*(A)$. Its limit is 0, so that $\varphi = 0$ follows. This contradicts the choice of φ and shows that δ has a dense image. This finishes the proof of the theorem. □

We finish this section by providing a general inversion formula, extending the one obtained in Lemma 9.5.6.

Lemma 9.5.9. *Let A be locally compact abelian group. Then $\mathcal{F} : (\mathrm{L}^1 \cap \mathrm{C}_0)(A) \to \mathrm{C}_0(\hat{A})$ has image in $\mathrm{C}_0^*(\hat{A})$ and extends to an isomorphism of Banach *-algebras $\mathcal{F} : \mathrm{C}_0^*(A) \to \mathrm{C}_0^*(\hat{A})$. Its inverse satisfies $(\mathcal{F}^{-1}\varphi)(x) = \hat{\varphi}(\delta_{x^{-1}})$.*

Proof. It is clear that $\mathcal{F}$ is a *-homomorphism from $\mathrm{C}^*(A)$ to $\mathrm{C}_0(\hat{A})$. We first prove that it restricts to a surjective isometry between $\mathrm{C}_0^*(A)$ and $\mathrm{C}_0^*(\hat{A})$ with the inverse map as stated in the lemma.

Let $B = \{f \in \mathrm{C}_0^*(A) \mid \hat{f} \in \mathrm{C_c}(\hat{A})\}$. By Lemma 9.5.6 and the Pontryagin Duality Theorem 9.5.8, we have for $f \in B$ that

$$\begin{aligned}\|f\|_0^* &= \max\{\|\hat{f}\|_{\hat{A}}, \|f\|_A\} \\ &= \max\{\|\hat{f}\|_{\hat{A}}, \|\hat{f}\|_{\hat{\hat{A}}}\} \\ &= \|\mathcal{F}(f)\|_0^*.\end{aligned}$$

So $\mathcal{F}$ extends to an isometry from $\overline{B}$ to a closed subspace of $\mathrm{C}_0^*(\hat{A})$. By Lemma 9.5.7, we have $\overline{B} = \mathrm{C}_0^*(A)$ and $\mathcal{F}(\mathrm{C}_0^*(A)) \subset \mathrm{C}_0^*(\hat{A})$ is dense. So $\mathcal{F} : \mathrm{C}_0^*(A) \to \mathrm{C}_0^*(\hat{A})$ is a surjective isometry.

By Lemma 9.5.6, an inverse for $\mathcal{F}$ on B is given by the composition

$$\mathrm{C}_0^*(\hat{A}) \xrightarrow{\mathcal{F}_{\hat{A}}} \mathrm{C}_0^*(\hat{\hat{A}}) \xrightarrow{\delta} \mathrm{C}_0^*(A) \xrightarrow{A \ni x \mapsto x^{-1}} \mathrm{C}_0^*(A).$$

By continuity, it follows that this composition is the inverse of $\mathcal{F}$, providing the formula in the statement. This finishes the proof of the lemma. □

Theorem 9.5.10 (Inversion formula). *Let A be a locally compact abelian group and $f \in \mathrm{L}^1(A)$ auch that $\hat{f} \in \mathrm{L}^1(\hat{A})$. Then $f \in \mathrm{C}_0(A)$ and*

$$f(x) = \hat{\hat{f}}(\delta_{x^{-1}})$$

for all $x \in A$.

Proof. We have $\hat{f} \in (\mathrm{L}^1 \cap \mathrm{C}_0)(\hat{A}) \subset \mathrm{C}_0^*(\hat{A})$. So $(\mathcal{F}^{-1}\hat{f}) \in \mathrm{C}_0^*(A)$ is well-defined and equals f due to injectivity of the Fourier transform. We conclude that

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \hat{\hat{f}}(\delta_{x^{-1}})$$

for all $x \in A$. □

The next theorem describes the relation between the Plancherel measure and evaluation at the identity in the most general context we are going to use. This was our aim, stated in equation (9.1) of Section 9.4.

Theorem 9.5.11. *Let A be a locally compact abelian group and $f \in \mathrm{C}_0^*(A)$ such that $\hat{f} \in \mathrm{L}^1(\hat{A})$. Then*

$$\int_{\hat{A}} \hat{f}(\chi)\mathrm{d}\chi = f(e)\,.$$

Proof. We observe that $\hat{f} \in (\mathrm{L}^1 \cap \mathrm{C}_0)(\hat{A})$ by assumption. Let $(\varphi_n)_n$ be a sequence in $\mathrm{C}_c(\hat{A})$ approximating $\hat{f}$. By Lemma 9.5.9, the functions $f_n = \mathcal{F}^{-1}(\varphi_n)$ are well-defined elements of $\mathrm{C}_0^*(A)$ and approximate f in $\mathrm{C}_0^*(A)$. It follows by Remark 9.4.15 that

$$f(e) = \lim f_n(e) = \lim \int_{\hat{A}} \hat{f}_n(\chi)\mathrm{d}\chi = \lim \int_{\hat{A}} \varphi_n(\chi)\mathrm{d}\chi = \int_{\hat{A}} \hat{f}(\chi)\mathrm{d}\chi\,.$$

This is what we had to show. □

9.6 The Plancherel theorem

We already saw in Lemma 9.5.9 the the Fourier transform induces isomorphisms between several functional analytic objects attached to a locally compact abelian group A and its Pontryagin dual $\hat{A}$. In this section, we show that it also induces an isomorphism in L^2.

Theorem 9.6.1 (Plancherel theorem). *Let A be a locally compact abelian group. Then the Fourier transform restricted to $(\mathrm{L}^1 \cap \mathrm{L}^2)(A)$ takes values in $(\mathrm{C}_0 \cap \mathrm{L}^2)(\hat{A})$ and extends to a unitary operator $\mathcal{F} : \mathrm{L}^2(A) \to \mathrm{L}^2(\hat{A})$.*

Proof. Let $f \in (\mathrm{L}^1 \cap \mathrm{L}^2)(A)$. Then $f * f^* \in (\mathrm{L}^1 \cap \mathrm{C}_0)(A)$ by Lemma 9.4.4 and $(f * f^*)(e) = \|f\|_2^2$. Further, $\mathcal{F}(f * f^*) = |\hat{f}|^2$ is a non-negative function. If $\varphi \in \mathrm{C}_c(\hat{A})$ satisfies $0 \leq \varphi \leq \mathcal{F}(f * f^*)$, then Lemma 9.4.10 implies that

$$\int_{\hat{A}} \varphi(\chi)\mathrm{d}\chi = \inf\{g(1) \mid g \in \mathrm{C}_c^*(A), \hat{g} \geq \varphi\} \leq (f * f^*)(e) = \|f\|_2^2 < \infty\,.$$

So $\mathcal{F}(f * f^*) \in \mathrm{L}^1(\hat{A})$ and Theorem 9.5.10 implies that

$$\begin{aligned} \|f\|_2^2 &= (f * f^*)(e) \\ &= (\mathcal{F} \circ \mathcal{F})(f * f^*)(\delta_e) \\ &= \mathcal{F}(|\hat{f}|^2)(\delta_e) \\ &= \int_{\hat{A}} |\hat{f}|^2(\chi)\delta_e(\chi)\mathrm{d}\chi \\ &= \int_{\hat{A}} |\hat{f}|^2(\chi)\mathrm{d}\chi \\ &= \|\hat{f}\|_2^2 . \end{aligned}$$

Since $\mathrm{C_c}(A) \subset (\mathrm{L}^1 \cap \mathrm{L}^2)(A) \subset \mathrm{L}^2(A)$ is dense, the Fourier transform extends to an isometry $\mathcal{F} : \mathrm{L}^2(A) \to \mathrm{L}^2(\hat{A})$. By Lemma 9.4.8, its image contains a dense subset of $\mathrm{C_c}(\hat{A})$. Hence it is surjective. □

Proposition 9.6.2. *Let $f, g \in (\mathrm{L}^1 \cap \mathrm{L}^2)(A)$. Then $f * g \in \mathrm{L}^1(A)$ and $\mathcal{F}(f * g) \in \mathrm{L}^1(A)$ and the inversion formula applies to $f * g$.*

Proof. Since $\mathrm{L}^1(A)$ is an algebra under the convolution product, we have $f * g \in \mathrm{L}^1(A)$. Further, the Plancherel Theorem 9.6.1 says that $\hat{f}, \hat{g} \in \mathrm{L}^2(\hat{A})$. So $\mathcal{F}(f * g) = \hat{f}\hat{g} \in \mathrm{L}^1(A)$ follows by the Cauchy-Schwarz inequality. Now Theorem 9.5.10 says that the inversion formula applies to $f * g$. □

Theorem 9.6.3. *Let A be a locally compact abelian group and $\mathcal{F} : \mathrm{L}^2(A) \to \mathrm{L}^2(\hat{A})$ the extension of the Fourier transform. Then $\operatorname{Ad}\mathcal{F} : \mathrm{C}^*(A) \to \mathrm{C}_0(\hat{A})$ is the Fourier transform on $\mathrm{C}^*(A)$. Further, we obtain an isomorphism $\operatorname{Ad}\mathcal{F} : \mathrm{L}(A) \to \mathrm{L}^\infty(\hat{A})$.*

Proof. By the Plancherel Theorem 9.6.1, the Fourier transform defines a unitary operator $\mathcal{F} : \mathrm{L}^2(A) \to \mathrm{L}^2(\hat{A})$. For $f \in \mathrm{C_c}(A) \subset \mathrm{C}^*(A)$ and $g \in \mathrm{C_c}(A) \subset \mathrm{L}^2(A)$, we have

$$\mathcal{F}\lambda(f)g = \mathcal{F}(f * g) = \hat{f}\mathcal{F}g .$$

This shows that $(\operatorname{Ad}\mathcal{F})(\lambda(f)) = \hat{f}$, where the latter acts by pointwise multiplication on $\mathrm{L}^2(\hat{A})$. So $\operatorname{Ad}\mathcal{F}$ agrees with the Fourier transform on the dense subalgebra $\mathrm{C_c}(A) \subset Cstar(A)$. This implies the first statement of the theorem.

Since $\operatorname{Ad}\mathcal{F} : \mathrm{C}^*(A) \to \mathrm{C}_0(\hat{A})$ is an isomorphism, we also obtain an isomorphism $\operatorname{Ad}\mathcal{F} : \overline{\mathrm{C}^*(A)}^{\mathrm{SOT}} \to \overline{\mathrm{C}_0(\hat{A})}^{\mathrm{SOT}}$. By Theorem 6.5.5, we have $\overline{\mathrm{C}^*(A)}^{\mathrm{SOT}} = \mathrm{L}(G)$. Further, the fact that $\mathrm{L}^\infty(\hat{A})$ lies in the strong closure of $\mathrm{C}_0(\hat{A})$ can be seen by approximating first indicator functions of compact subsets, then of measurable subsets and finally arbitrary functions in $\mathrm{L}^\infty(\hat{A})$. The fact that the strong closure of $\mathrm{C}_0(\hat{A})$ equals $\mathrm{L}^\infty(\hat{A})$ then follows from the fact that the latter is equal to its commutant $\mathrm{L}^\infty(\hat{A})'$ and the bicommutant theorem 6.5.2. This finishes the proof of the theorem. □

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