Homotopy Invariance of Relative Singular homology

Francesco Fournier Facio. March 25, 2018

The aim of this presentation is to show that relative singular homology satisfies the third Eilenberg-Steenrod axiom for homology: homotopy invariance. More precisely, we set out to prove the following theorem.

Theorem 1. *Let $f, g : (X, A) \to (Y, B)$ be morphisms between pairs of topological spaces. If f and g are homotopic, then they induce the same morphisms on relative singular homologies.*

The algebraic tool that will allow us to prove this theorem is that of chain homotopy. Let C and D be two chain complexes. An easy way to create a chain map $f : C \to D$ is to chose random morphisms $s_n : C_n \to D_{n+1}$, which gives us two different maps $C_n \to D_{n+1} \to D_n$ and $C_n \to C_{n-1} \to D_n$. Adding them we define $f_n = d^D_{n+1} s_n + s_{n-1} d^C_n : C_n \to D_n$. One can then easily check that f is a chain map.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[dl, "s_n"'] \arrow[d, dashed, "f_n"] & C_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 2. A chain map $f : C \to D$ is *null homotopic* if there are morphisms $s_n : C_n \to D_{n+1}$ such that $f_n = d^D_{n+1} s_n + s_{n-1} d^C_n$. Two chain maps $f, g : C \to D$ are *chain homotopic* if their difference $f - g$ is null homotopic. The corresponding $\{s_n\}$ form a *chain homotopy* from f to g.

Two chain homotopic maps are "similar" in a sense that will be made rigorous shortly, and is intuitively analogous to that of homotopic continuous maps: we can transform one into the other using a chain homotopy. While it is fairly easy to generate null homotopic chain maps, these maps are particularly simple: a null homotopic map is, by definition, a chain map that is chain homotopic to the zero map. However, we can use this fact to get an important criterion for when two chain maps induce the same morphisms on homologies.

Lemma 3. *If f and g are chain homotopic, then they induce the same morphisms on homologies.*

Proof. By considering $f - g$, it suffices to show that if f is null homotopic, then $H_n(f) : H_n(C) \to H_n(D)$ is the zero morphism. Suppose that there exist maps $s_n : C_n \to D_n + 1$ such that $f_n = d^D_{n+1} s_n + s_{n-1} d^C_n$. Let $x \in \ker d^C_n$. In what follows, square brackets denote the class of an element in the respective homology group.

$$H_n(f)([x]) = [f_n(x)] = [d^D_{n+1} s_n(x) + s_{n-1} \underbrace{d^C_n(x)}_{=0}] = [\underbrace{d^D_{n+1}(s_n(x))}_{\in \operatorname{im} d^D_{n+1}}] = [0].$$

$\square$

Now that we have this criterion, we are ready to prove a non-relative version of theorem 1.

Theorem 4. *Let $f, g : X \to Y$ be continuous maps between topological spaces. If f and g are homotopic, then they induce the same morphisms on singular homologies.*

Proof. Let $F : X \times I \to Y$ be a homotopy from f to g, and let f^* and g^* denote the induced chain maps $C(X) \to C(Y)$. We will use F to provide diagonal maps $C_n(X) \to C_{n+1}(Y)$, and then show that these form a chain homotopy from g^* to f^*.
Let $\sigma : \Delta^n \to X$ be an element of $C_n(X)$. Then we have a map $F_\sigma := F \circ (\sigma, id_I) : \Delta^n \times I \to X \times I \to Y$. Let $[v_0, \dots, v_n] := \Delta^n \times \{0\}$ and $[w_0, \dots, w_n] := \Delta^{n+1} \times \{1\}$. These two n-simplices verify: $F_\sigma|_{\Delta^n \times \{0\}} : x \mapsto F(\sigma(x), 0) = f(\sigma(x)) = f^*(\sigma)(x)$ and similarly $F_\sigma|_{\Delta^n \times \{1\}} = g^*(\sigma)$. Note that $\Delta^n \times I$ has $(n+1)$ dimensions. Most importantly, for all $i = 0, \dots, n$ the region $[v_0, \dots, v_i, w_i, \dots w_n] \subset \Delta^n \times I$

is an $(n+1)$ simplex, and so $F^i_\sigma := F_\sigma|_{[v_0,\dots,v_i,w_i,\dots w_n]} \in C_{n+1}(Y)$. This allows us to define the *prism operators*:

$$P_n : C_n(X) \to C_{n+1}(Y) : \sigma \mapsto \sum_{i=0}^{n} (-1)^i \cdot F^i_\sigma.$$

We claim that $d^Y_{n+1} P_n = g^*_n - f^*_n - P_{n-1} d^X_n$. In the following calculations, we drop the subscripts for clarity. We start with:

$$d^Y P(\sigma) = d^Y \Big(\sum_i (-1)^i \cdot F^i_\sigma \Big) = \sum_i (-1)^i \cdot d^Y(F^i_\sigma) = \sum_i (-1)^i \cdot \Big(\sum_{j \le i} (-1)^j \cdot F_\sigma|_{[v_0,\dots,\hat{v}_j,\dots,w_n]} +$$
$$+ \sum_{j \ge i} (-1)^{j+1} \cdot F_\sigma|_{[v_0,\dots,\hat{w}_j,\dots,w_n]} \Big) =: S_{(i=j)} + S_{(i \ne j)};$$

where $S_{(i=j)}$ denotes the terms for which $i = j$ and similarly for $S_{(i \ne j)}$. We will calculate these two expressions separately. On one hand:

$$S_{(i=j)} = \sum_i F_\sigma|_{[v_0,\dots,\hat{v}_i,w_i,\dots,w_n]} - \sum_i F_\sigma|_{[v_0,\dots,v_i,\hat{w}_i,\dots,w_n]} = F_\sigma|_{[\hat{v}_0,w_0,\dots,w_n]} - F_\sigma|_{[v_0,\dots,v_n,\hat{w}_n]} +$$
$$+ \sum_{i \ne 0} \big(\underbrace{F_\sigma|_{[v_0,\dots,\hat{v}_i,w_i,\dots,w_n]} - F_\sigma|_{[v_0,\dots,v_{i-1},\widehat{w_{i-1}},\dots,w_n]}}_{=0} \big) = F_\sigma|_{\Delta^n \times \{1\}} - F_\sigma|_{\Delta^n \times \{0\}} = g^*(\sigma) - f^*(\sigma).$$

On the other hand:

$$Pd^X(\sigma) = P\Big(\sum_j (-1)^j \cdot \sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_n]} \Big) = \sum_j (-1)^j \cdot P(\sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}) = \sum_j (-1)^j \cdot$$
$$\cdot \Big(\sum_{i<j} (-1)^i \cdot F_\sigma|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots w_n]} + \sum_{i>j} (-1)^{i-1} \cdot F_\sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots w_n]} \Big) = -S_{(i \ne j)}.$$

Thus $d^Y P = S_{(i=j)} + S_{(i \ne j)} = g^* - f^* - Pd^X$, which implies by definition that P is a chain homotopy from g^* to f^*. By lemma 3, f^* and g^* induce the same morphisms on homologies, so the same goes for f and g. □

This proves that singular homology is homotopy invariant. However, in order to satisfy the Eilenberg-Steenrod axions, we have to prove this for relative singular homology. Luckily, almost all the work has been done in the proof of theorem 4.

Proof of theorem 1. Let $f, g : (X, A) \to (Y, B)$ be morphisms between pairs of topological spaces. Let F be a homotopy from f to g in the category of pairs of topological spaces. This means that $F : X \times I \to Y$ is a homotopy from f to g, seen as continuous maps, such that for all $t \in I$, the map $F(-,t) : X \to Y$ takes A to B. Then the prism operator $P_n : C_n(X) \to C_{n+1}(Y)$ introduced in the proof of theorem 4 takes $C_n(A)$ to $C_{n+1}(B)$, so the induced map on the quotient groups $\bar{P}_n : C_n(X, A) \to C_{n+1}(Y, B)$is well-defined. Finally, the equality $d^Y_{n+1} P_n = g^*_n - f^*_n - P_{n+1} d^X_n$ is preserved after taking the quotient, so the maps induced by f and g on the relative chain complexes are chain homotopic. Once again, lemma 3 implies that f and g induce the same morphisms on relative homologies. □

We conclude that, as far as homotopy invariance is concerned, relative singular homology behaves like a homology theory in the sense of the Eilenberg-Steenrod axioms.

Resources

[1] A. Hatcher: *Algebraic Topology*: `https://www.math.cornell.edu/~hatcher/AT/ATpage.html`. Theorem 2.10, p.111 and Proposition 2.19, p.118.

[2] C. A. Weibel: *An Introduction to Homological Algebra*. Cambridge University Press, 1994. Chapter 1.4.