Local degree

Note that a choice of isomorphism $H_n(S^n) \cong \mathbb{Z}$ and a point $x \in S^n$ ~~induce on an isomorphism~~ with a neighborhood $U \ni x$,

H^n

induce an isomorphism using excision

$$H_n(U, U \setminus \{x\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x\}) \xrightarrow{\cong} H_n(S^n) \cong \mathbb{Z}$$

since the long exact sequence shows

$$\begin{array}{ccccccc} H_n(S^n \setminus \{x\}) & \to & H_n(S^n) & \xrightarrow{\cong} & H_n(S^n, S^n \setminus \{x\}) & \to & H_{n-1}(S^n \setminus \{x\}) \\ \| & & \wr\| & & & & \| \\ 0 & & \mathbb{Z} & & & & 0 \end{array}$$

~~Proposition~~

5.13 **Definition**

Let $f: S^n \to S^n$ be cont and $y \in S^n$ st $f^{-1}(y)$ is finite. Let $U_1, \dots, U_m$ be disjoint neighborhoods of the elements $x_1, \dots, x_m \in f^{-1}(y)$. Fix an isomorphism $H_n(S^n) \cong \mathbb{Z}$. Then the local degree of f at $x_i \in f^{-1}(y)$ is

$$\begin{array}{ccc} H_n(U_i, U_i \setminus \{x_i\}) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus \{y\}) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z} & \xrightarrow{\deg f|x_i} & \mathbb{Z} \end{array}$$

5.14 Proposition

Let $f: S^n \to S^n$ and $y \in S^n$ st $f^{-1}(y)$ is finite. Let $x_1, \dots, x_m$ be the preimages of y under f. Then

$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i} .$$

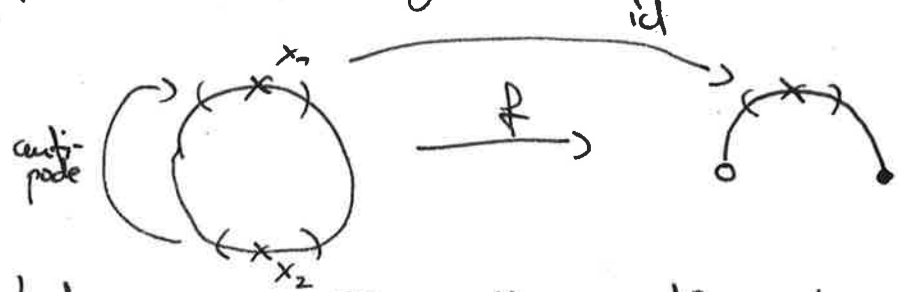
Without proof

5.15 Example

We calculate the degree of ~~$S^n \to \mathbb{R}P^n \to S^n$~~

$$f: S^n \to \mathbb{R}P^n \to \mathbb{R}P^n / \mathbb{R}P^{n-1} \to S^n$$

We make use of the fact that the antipode of S^n has degree $(-1)^n$ (exercise).



Let $y \in S^n$ with exactly two preimages x_1, x_2. under f. There are neighbourhoods $U_i \ni x_i$ st. $f|_{U_1}$ is the identity and $f|_{U_2} = f|_{U_1} \circ (-id)$.

So
$$\deg f|_{x_1} = 1$$

$$\deg f|_{x_2} = \cancel{1} \deg f|_{x_1} \cdot \deg(-id)$$
$$= 1 \cdot (-1)^n$$

So $\deg f = 1 + (-1)^{n+1}$.

Let us use this ~~ex~~ to calcule $H_k^{cell}(\mathbb{RP}^n)$

$$\begin{array}{ccccccccccc} & & \text{one 0-cell} & & & & & & \text{2 if } n \text{ even, 0 if } n \text{ odd} & & \\ 0 & \leftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{1+(-1)^{2}=2} & \mathbb{Z} & \xleftarrow{0} & \cdots & \leftarrow \mathbb{Z} & \leftarrow 0 \\ & & \cong & & \cong & & \cong & & & \cong & \\ & & H_0^{cell}(\mathbb{RP}^n) & & H_1^{cell}(\mathbb{RP}^n) & & H_2^{cell}(\mathbb{RP}^n) & & & H_n^{cell}(\mathbb{RP}^n) & \end{array}$$

So $$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } k \text{ odd}, 0 \leq k < n \\ 0 & \text{otherwise} \end{cases}$$

5.16 <u>Theorem</u>

H_*^{cell} is a homology theory on the category of CW-pairs (pairs of the homotopy type of a CW-pair).

5.17 <u>Theorem</u>

There is a unique homology theory on the category of pairs of the homotopy type of a CW-pair.