Exercises on Homology and Cohomology

Spring term 2018, Sheet 4

Hand in before 10 o'clock on 19th March 2018
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Exercise 1 (easy)
An extension of groups is a diagram

$$1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

such that

- ι is injective,
- π is surjective,
- $\ker \pi = \operatorname{im} \iota$.

Informally, the image of each morphism equals the kernel of the next morphism. The extension above is called split if there is a morphism $s : Q \to G$ such that $\pi \circ s = \mathrm{id}_Q$.

(i) Show that an extension of groups $1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$ is split if and only if $G = N \rtimes Q$ is a semi-direct product.

(ii) Provide an example of a non-split extension.

(iii) Show that every extension $1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} \mathbb{Z} \to 1$ is split.

Exercise 2 (medium)
Denote by $\tilde{\mathrm{H}}_n(X)$ the n-th reduced singular homology group of a topological space X.

(i) Prove that there is a short exact sequence $0 \to \tilde{\mathrm{H}}_0(X) \to \mathrm{H}_0(X) \to \mathbb{Z} \to 0$ and that

$$\mathrm{H}_n(X) \cong \begin{cases} \tilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & n = 0 \\ \tilde{\mathrm{H}}_n(X) \,. \end{cases}$$

(ii) Show that H_n and $\tilde{\mathrm{H}}_n$ are functors.

Exercise 3 (difficult)
Let $(\mathrm{H}_n)_{n \in \mathbb{N}}$ be a homology theory in the sense of the Eilenberg-Steenrod axioms with coefficients A. Compute $\mathrm{H}_n(\mathrm{S}^1)$.