Exercises on Homology and Cohomology

Spring term 2018, Sheet 7

Hand in before 10 o'clock on 16th April 2018
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Exercise 1 (easy)
In this exercise we calculate the homology of spheres of positive dimension

$$\mathrm{H}_k(\mathrm{S}^n) = \begin{cases} \mathbb{Z} & \text{if } k \in \{0, n\} \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that there are open contractible sets $U, V \subset \mathrm{S}^n$ such that

- $\mathrm{S}^n = U \cup V$, and
- $U \cap V \sim_{\mathrm{h}} \mathrm{S}^{n-1}$ are homotopy equivalent.

(ii) Use the Meyer-Vietoris exact sequence of Exercise 2 to inductively calculate the homology of S^n.

Exercise 2 (medium)
In this exercise we provide an alternative and frequently useful way to apply excision in homology.

(i) *(Algebraic Meyer-Vietoris sequence)* Consider a commutative diagram of R-moudles whose rows are exact such that the maps $(\varphi_n)_n$ are isomorphisms:

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & C_{n+1} & \xrightarrow{\partial^C_{n+1}} & C''_n & \xrightarrow{i_n} & C'_n & \xrightarrow{p_n} & C_n & \xrightarrow{\partial^C_n} & C''_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi''_n} & & \downarrow{\scriptstyle \varphi'_n} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi''_{n-1}} & & \\
\dots & \longrightarrow & D_{n+1} & \xrightarrow{\partial^D_{n+1}} & D''_n & \xrightarrow{j_n} & D'_n & \xrightarrow{q_n} & D_n & \xrightarrow{\partial^D_n} & D''_{n-1} & \longrightarrow & \dots
\end{array}$$

Show that there is an exact sequence

$$\dots \longrightarrow C''_n \xrightarrow{(i_n, \varphi''_n)} C'_n \oplus D''_n \xrightarrow{\varphi_n - j_n} D'_n \xrightarrow{\delta_n} C''_{n-1} \longrightarrow \dots$$

where $\delta_n = \partial_n \circ \varphi_n^{-1} \circ q_n : D'_n \to C''_{n-1}$.

(ii) *(Topological Meyer-Vietoris sequence)* Let $(\mathrm{H}_n)_{n \in \mathbb{N}}$ be a homology theory and let $X = U^\circ \cup V^\circ$ be a cover a topological space X by the interior of two subsets $U, V \subset X$. Use excision to apply the algebraic Meyer-Vietoris sequence to the diagram of long exact sequences induced by the inclusion $(U, U \cap V) \hookrightarrow (X, V)$

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathrm{H}_{n+1}(U, U \cap V) & \longrightarrow & \mathrm{H}_n(U \cap V) & \longrightarrow & \mathrm{H}_n(U) & \longrightarrow & \mathrm{H}_n(U, U \cap V) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{H}_{n+1}(X, V) & \longrightarrow & \mathrm{H}_n(V) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, V) & \longrightarrow & \dots
\end{array}$$

to obtain an exact sequence

$$\dots \longrightarrow \mathrm{H}_n(U \cap V) \longrightarrow \mathrm{H}_n(U) \oplus \mathrm{H}_n(V) \longrightarrow \mathrm{H}_n(X) \longrightarrow \mathrm{H}_{n-1}(U \cap V) \longrightarrow \dots$$

Exercise 3 (difficult)

In this exercise we provide a way to calculate relative homology. We say that a pair (X, A) has the homotopy extension property (hep), if whenever $\varphi_t : A \to Z$, $t \in [0, 1]$ is a homotopy of continuous maps and $\psi_0 : X \to Z$ is an extension of φ_0, then there is some homotopy of continuous maps $\psi_t : X \to Z$ extending $(\varphi_t)_{t\in[0,1]}$. We will see examples of pairs with the homotopy extension property later.

Let (X, A) be a pair of topological spaces with the hep. Show that

$$\mathrm{H}_*(X, A) \cong \mathrm{H}_*(X/A, \{A\}) \cong \tilde{\mathrm{H}}_*(X/A)$$

induced by the quotient map $(X, A) \to (X/A, \{A\})$ and the identification $\tilde{H}_*(X) \cong \mathrm{H}_*(X, \{pt\})$.