Exercises on Homology and Cohomology

Spring term 2018, Sheet 8

Hand in before 10 o'clock on 23rd April 2018
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Exercise 1 (easy)
In this exercise we calculate the cellular homology of the complex projective plane $\mathbb{C}\mathrm{P}^n$.

(i) Show that $\mathbb{C}\mathrm{P}^n$ admits a structure or a CW-complex with exactly one cell in degree $0 \leq 2k \leq 2n$.

(ii) Calculate $\mathrm{H}_*^{\mathrm{cell}}(\mathbb{C}\mathrm{P}^n)$.

Exercise 2 (medium)
In this exercise we investigate the degree of the antipode on a sphere.

(i) Fix some $i \in \{1, \ldots, n+1\}$ and denote by $f : \mathrm{S}^n \to \mathrm{S}^n$ the reflection about the i-th hyperplane, that is $f(v_1, \ldots, v_n) = (v_1, \ldots, -v_i, \ldots, v_n)$. Find a Δ-complex structure on the sphere S^n, which has exactly 2-simplices of dimension n, which are permuted by f.

(ii) Calculate the degree of f.

(iii) Conclude that the degree of the antipode of S^n is $(-1)^{n+1}$.

Exercise 3 (difficult)
In this exercise we provide examples of spaces with the homotopy extension property (hep) featuring in Exercise 3 of Sheet 7. A pair of topological spaces (X, A) is called a CW-pair, if there is the structure of a CW-complex on X that turns $A \subset X$ into a subcomplex.

(i) Show that a pair of space (X, A) has the hep if and only if there is a retract

$$r : X \times I \to X \times \{0\} \cup A \times I \,,$$

that a continuous map that satisfies $r|_{X \times \{0\} \cup A \times I} = \mathrm{id}$.

(ii) Show that $\mathrm{S}^{n-1} \subset \mathrm{D}^n$ has the hep.

(iii) Given a pushout

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & Y \end{array}$$

where $A \hookrightarrow X$ is an inclusion, show that $B \to Y$ is an inclusion. Further, if $A \subset X$ has the hep, then also $B \subset Y$ has the hep.

(iv) Let I be a well-ordered set with minimal element $0 \in I$ and $X = \cup_{i \in I} X_i$ a directed union of spaces. Assume that $X_i \subset X_j$ has the hep for every $i \leq j$ and show that $X_0 \subset X$ has the hep.

(v) Conclude that every CW-pair has the hep.