Exercises on Homology and Cohomology

Spring term 2018, Sheet 9

Hand in before 10 o'clock on 30th April 2018
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Exercise 1 (easy)

In this exercise we illustrate that the ring structure in cohomology can distinguish some spaces, where usual (co)homology can not.

(i) A ring S is graded, if there it is a direct sum of abelian groups $S = \bigoplus_{k \in \mathbb{N}} S^k$ such that multiplication restricts to maps $S^k \times S^l \to S^{k+l}$. Let (X, A) be a pair of spaces and R a ring. Show that the cup product defines a graded ring structure on $\mathrm{H}^*(X, A; R) = \bigoplus_{k \in \mathbb{N}} \mathrm{H}^k(X, A; R)$

(ii) Calculate the cohomology ring $\mathrm{H}^*(\mathrm{S}^n; R) \cong R[x]/x^2$, where x has degree n.

(iii) We admit that $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ as ring is isomorphic with $\mathbb{Z}[x]/x^{n+1}$, where $\mathbb{Z}[x]$ denotes the polynomial ring in one variable. Show that that $\mathbb{C}P^2$ is not homotopy equivalent with $\mathrm{S}^2 \vee \mathrm{S}^4$, using the result of Exercise 2.

Exercise 2 (medium)

This exercise starts from the Eilenberg-Steenrod axioms, and in particular additivity: singular cohomology with arbitrary coefficients satisfies additivity in the sense that for any family of space $(X_i)_{i \in I}$ the natural inclusions $\iota_i : X_i \to \coprod X_i$ induce an isomorphism in cohomology

$$\mathrm{H}^*(\coprod_{i \in I} X_i; M) \xrightarrow{\prod \iota_i} \prod_{i \in I} \mathrm{H}^*(X_i; M) \,.$$

Remark that on the right side, we see a product of groups instead of a direct sum, which makes a difference only for infinite index sets I.

Next recall the wedge sum of two pointed spaces (X, x) and (Y, y) defined as $X \vee Y = X \coprod Y / x = y$. This construction is interpreted as taking the disjoint union of X and Y and identifying their base point. More generally, for a family $(X_i, x_i)_{i \in I}$ of space the wedge sum is defined as

$$\bigvee_{i \in I} X_i = \coprod_{i \in I} X_i / \coprod_{i \in I} \{x_i\} \,.$$

(i) Show that for reduced cohomology of pointed spaces, defined as $\tilde{\mathrm{H}}^*(X; M) = \mathrm{H}^*(X, \{x\}; M)$ the following additivity formula holds if all (X_i, x_i) are pointed CW-complexes

$$\tilde{\mathrm{H}}^*(\bigvee_{i \in I} X_i; M) \xrightarrow{\prod \iota_i} \prod_{i \in I} \tilde{\mathrm{H}}^*(X_i; M) \,.$$

The assumption that (X_i, x_i) is a pointed CW-complex, implies that the quotient map $\coprod_{i \in I} X_i \to \coprod_{i \in I} X_i / \coprod_{i \in I} x_i$ induces an isomorphism

$$\mathrm{H}^*(\coprod_{i \in I} X_i, \coprod x_i; M) \to \tilde{\mathrm{H}}^*(\coprod_{i \in I} X_i; M)$$

You may use this fact without providing a proof.

(ii) Assume that R is a ring and show that the isomorphism provided by additivity in reduced cohomology $\tilde{\mathrm{H}}^*(X;R)$ and non-reduced cohomology $\mathrm{H}^*(X;R)$ is a graded ring isomorphism.

(iii) Calculate the ring $\tilde{\mathrm{H}}^*(\mathrm{S}^2 \vee \mathrm{S}^4)$, using the result of Exercise 1 (ii).

Exercise 3 (difficult)

Assume the following statement: chain map between chain complexes $\varphi : C_* \to D_*$ of free abelian groups that induces an isomorphism in homology also induces an isomorphism in cohomology for arbitrary coefficients through the map $\varphi^* : \mathrm{Hom}(D_*, M) \to \mathrm{Hom}(C_*, M)$.

(i) Formulate Eilenberg-Steenrod axioms for cohomology.

(ii) Show that singular cohomology $\mathrm{H}^*(-,-;M)$ is a cohomology theory with coefficient group M.