Exercises on Lie groups

Spring term 2018, Sheet 9

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Exercise 1

In this exercise we investigate the relation between kernels, normal subgroups and Lie ideals.

(i) Show that the closed normal subgroups of a Lie group G are exactly the kernels of Lie homomorphisms $G \to H$.

(ii) Show that the Lie ideals of a Lie algebra $\mathfrak{g}$ are exactly the kernels of Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{h}$.

(iii) Let $\pi : G \to H$ be a Lie group homomorphism and write $N = \ker \pi$. Denote by $\pi_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ the derivative of π and denote its kernel by $\mathfrak{n} = \ker \pi_*$. Show that $\mathrm{Lie}(N) = \mathfrak{n}$.

Exercise 2

In this exercise we compare the internal and the external semi-direct product of Lie algebras.

(i) Let $\mathfrak{g}$ be a Lie algebra and denote by

$$\mathrm{Der}(\mathfrak{g}) = \{\delta : \mathfrak{g} \to \mathfrak{g} \mid [\delta(X), \delta(Y)] = [\delta(X), Y] + [X, \delta(Y)]\}$$

its derivation Lie algebra. Show that $\mathrm{Der}(\mathfrak{g})$ is a Lie algebra with respect to the commutator bracket

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$$

(ii) Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $\alpha : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ a Lie algebra homomorphism. Show that the vector space $\mathfrak{h} \oplus \mathfrak{g}$ equipped with the bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, \alpha_{Y_1}(X_2)] - [X_2, \alpha_{Y_2}(X_1)], [Y_1, Y_2])$$

is a Lie algebra, which will be the semi-direct product Lie algebra $\mathfrak{h} \rtimes \mathfrak{g}$. Further show that the natural injections $\mathfrak{g}, \mathfrak{h} \hookrightarrow \mathfrak{h} \rtimes \mathfrak{g}$ are Lie algebra homomorphisms and that $\mathfrak{h} \trianglelefteq \mathfrak{h} \rtimes \mathfrak{g}$ is an ideal.

(iii) Let $\mathfrak{n} \trianglelefteq \mathfrak{g}$ be an ideal of a Lie algebra and assume that there is some Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as vector spaces. Show that there is a natural Lie algebra homomorphism $\mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$ and a unique Lie algebra homomorphism $\mathfrak{n} \rtimes \mathfrak{h} \to \mathfrak{g}$ restricting to the inclusion maps of $\mathfrak{n}$ and $\mathfrak{h}$.

Exercise 3

We inductively define the upper central series of a group G by putting $Z_1 = \mathcal{Z}(G)$ equal to the centre and

$$Z_n = \{g \in G \mid gZ_{n-1} \in \mathcal{Z}(G/Z_{n-1})\}.$$

A group is called nilpotent, if there is some $n \in \mathbb{N}$ such that $Z_n = G$.

(i) Show that every nilpotent group is solvable.

(ii) Decide which of the following group is solvable and nilpotent:

- $\mathbb{C}^2 \rtimes_\alpha \mathbb{R}$ where $\alpha_t(z_1, z_2) = (e^{2\pi i t} z_1, e^{2\pi i \theta t} z_2)$ for some fixed irrational $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- The Heisenberg group $\mathrm{Heis}(\mathbb{R})$.
- The group of invertible upper triangular matrices

$$\mathrm{IT}(n) = \begin{pmatrix} * & * & * & \dots & * \\ 0 & * & * & \dots & * \\ \dots & & & & \\ \dots & & & & \\ \dots & & & * & * \\ 0 & \dots & & 0 & * \end{pmatrix}$$

- The group of unipotent upper triangular matrices

$$\mathrm{UT}(n) = \begin{pmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ \dots & & & & \\ \dots & & & & \\ \dots & & & 1 & * \\ 0 & \dots & & 0 & 1 \end{pmatrix}$$

Exercise 4.
Show that $\mathrm{SU}(2)$ is the set of its commutators,

$$\mathrm{SU}(2) = \{[g, h] \mid g, h \in \mathrm{SU}(2)\}.$$