Exercises on Lie groups

Spring term 2018, Sheet 2

Hand in before 10 o'clock on 2nd March 2018
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Notice
The lecture of Monday 5th March will be not take place. Instead, there will be a lecture on Fridaz 2nd March between 13.15 and 15 o'clock in room MA A3 31.

Exercise 1.
Show that a manifold is connected if and only if it is path connected.

Exercise 2.
Recall that an algebra A is called associative if and only if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds for all $a, b, c \in A$. Further, a Lie algebra A is called 2-step nilpotent if $[[a, b], c] = 0$ for all $a, b, c \in A$. One can show that the Lie algebra of a Lie group is 2-step nilpotent if and only if the Lie group is 2-step nilpotent in the group theoretical sense.

Show that a Lie algebra is associative if and only if it is 2-step nilpotent.

Exercise 3.
Prove that $\mathrm{Lie}(\mathbb{R}^n) = \mathrm{span}\{\frac{\partial}{\partial x_i} \mid i \in \{1, \dots, n\}\}$.

Exercise 4.
Let A be an (associative) R-algebra. Show that

$$[a, b] = ab - ba$$

introduces a Lie algebra structure on A. Deduce that $\mathcal{D}^1(M)$ and $\mathcal{D}^1(M, p)$ are Lie algebras for all differentiable manifolds M and all $p \in M$.

Exercise 5.
Let G, H be Lie groups. Use the identifications $\mathrm{Lie}(G) \cong \mathcal{D}^1(G, e)$ and $\mathrm{Lie}(H) \cong \mathcal{D}^1(H, e)$ to show that $\mathrm{Lie}(G \times H) \cong \mathrm{Lie}(G) \times \mathrm{Lie}(H)$ as Lie algebras.