Exercises on Lie groups

Spring term 2018, Sheet 4

Hand in before 10 o'clock on 16th March 2018
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Exercise 1.
Let $H \leq G$ be a closed normal Lie subgroup of a Lie group. Show that G/H is a Lie group.

Exercise 2.
In this exercise we give an example of a Lie group which is not a matrix Lie group.

The real Heisenberg group is

$$\mathrm{Heis}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{M}_3(\mathbb{R}) \right\}.$$

(i) (a) Determine the centre of $\mathrm{Heis}(\mathbb{R})$.

(b) Calculate the Lie algebra $\mathfrak{h}$ of $\mathrm{Heis}(\mathbb{R})$ as a Lie subalgebra of $\mathfrak{gl}(3, \mathbb{R})$.

(c) Consider the element

$$C \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}$$

and let $\pi : \mathfrak{h} \to \mathfrak{gl}(n, \mathbb{R})$ be any Lie representation. Using the fact that the trace of a commutator is 0, show that $\pi(C)$ is a nilpotent matrix.

(ii) Write

$$c_t \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(tC).$$

and let

$$G = \mathrm{Heis}(\mathbb{R}) / \langle c_n \mid n \in \mathbb{Z} \rangle.$$

(a) Show that G is a Lie group.

(b) Let $\pi : G \to \mathrm{GL}(n, \mathbb{R})$ be a Lie representation of G. Show that $\ker \pi \supset \{c_{t+\mathbb{Z}} \mid t \in \mathbb{R}\}$. Use the identification

$$\pi(c_t) = \pi(\exp(tC)) = \exp(t \mathrm{d}\pi(C))$$

and the fact that $\mathrm{d}\pi(C)$ is a nilpotent matrix.

(c) Generalising the initial definition of matrix Lie groups as closed subgroups of $\mathrm{GL}(n, \mathbb{K})$, we say that a Lie group is a matrix Lie group if it admits a faithful closed differentiable representation into $\mathrm{GL}(n, \mathbb{C})$. Show that G is not a matrix Lie group.

Exercise 3

(i) Denote by $\exp$ the matrix exponential and show that $\det \exp A = \exp \mathrm{tr} A$ for all $A \in \mathrm{M}_n(\mathbb{C})$.

(ii) Determine the Lie Algebra of $\mathrm{SL}(n, \mathbb{K})$, denoted $\mathfrak{sl}(n, \mathbb{K})$, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.