Exercises on Lie groups

Spring term 2018, Sheet 7

Hand in before 10 o'clock on 20th April 2018
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Exercise 1.
In this exercise we see that the correspondence between connected Lie subgroups and Lie subalgebras does not suffice to detect closed subgroups.

(i) Show that the quotient map map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$ induces an isomorphism of Lie algberas

(ii) Determine $\mathrm{Lie}(\mathbb{T}^2) = \mathrm{Lie}(\mathbb{R}^2)$.

(iii) Characterise those Lie subalgebras of $\mathrm{Lie}(\mathbb{T}^2)$ that correspond to closed subgroups.

(iv) Characterise those Lie subalgebras of $\mathrm{Lie}(\mathbb{R}^2)$ that correspond to closed subgroups.

Exercise 2.
Let G be a connected Lie group. Show that $\mathrm{Aut}(G)$ is a Lie group.

Exercise 3.
Let $(H_i)_{i \in I}$ be a family of closed Lie subgroups of a Lie group G whose Lie algebras are denoted $(\mathfrak{h}_i)_{i \in I}$. Without using the characterisation of closed subgroups in Lie groups, show that $H = \bigcap_{i \in I}$ is a closed Lie subgroup of G whose Lie algebra is $\bigcap_{i \in I} \mathfrak{h}_i$.

Exercise 4.
Let G be a connected Lie group. Show that for every closed subgroup $H \leq G$ there is a differentiable manifold M, some point $p \in M$ and a continuous action $G \curvearrowright M$ by diffeomorphisms such that $H = G_p = \{g \in G \mid gp = p\}$.

Exercise 5.
Recall the notion of a differential operator on a differential manifold M: this is a linear map $D : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ such that

- for all $f \in \mathrm{C}^\infty(M)$, we have $\mathrm{supp}\, Df \subset \mathrm{supp}\, f$.
- for every point $p \in M$ there are local coordinates (U, φ) of M at p such that $\varphi_*^{-1} \circ D \circ \varphi_* : \mathrm{C}^\infty(\varphi(U)) \to \mathrm{C}^\infty(\varphi(U))$ is a differential operator on the open subset $\varphi(U) \subset \mathbb{R}^n$.

Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Show that the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ acts through G-equivariant differential operators on G and that every G-equivariant differential operator on G arises this way.