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C*-SIMPLICITY [after Breuillard, Haagerup, Kalantar, Kennedy and Ozawa]

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INTRODUCTION

Associated with a discrete group G there is the reduced group C*-algebra, defined as the closure of the complex group ring $\mathbb{C}G$ acting on the Hilbert space $\ell^2(G)$ of square summable squences on G as bounded operators by left convolution $u_g\delta_h = \delta_{gh}$. We denote the reduced group C*-algebra by $\mathrm{C}^*_{\mathrm{red}}(G) = \overline{\mathbb{C}G}^{\|\cdot\|} \subset \mathcal{B}(\ell^2 G)$. This C*-algebra naturally relates to unitary representation theory of G through the notion of weak containment of representations [17]. A discrete group G is called C*-simple if $\mathrm{C}^*_{\mathrm{red}}(G)$ is simple as a C*-algebra, that is every two-sided closed ideal is trivial, which translates to the property that every weakly regular unitary representation of G is automatically weakly equivalent to its regular representation. Immediately from these definitions it is clear that results on C*-simplicity can provide interesting examples of C*-algebras and they help to provide norm estimates for operators in unitary representations. Moreover, C*-simplicity can be considered a strong form of non-amenability of groups. The original motivation to study C*-simplicity was purely operator algebraic: as Pierre de la Harpe reports in [13, p. 13], Powers was motivated to prove simplicity of $\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_2)$ by a question due to Kaplansky: is every unital simple C*-algebra generated by its projections? Kadison's suggestion that $\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_2)$ might solve this question in the negative, led to a proof of its simplicity within two weeks, already in 1967. However, this result [49] was only published in 1975 and it took until 1984 when Pimser–Voiculescu could prove absence of non-trivial projections in $\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_2)$. One year earlier, Blackadar had solved Kaplansky's question in the negative by completely different methods.

After Powers published his result in 1975 until 2014, research on C*-simplicity was dominated by combinatorial methods which were formalised by Pierre de la Harpe [12, p. 232] in terms of the Powers property later followed by numerous weakenings and variations. Early on it was known that normal amenable subgroups $N \trianglelefteq G$ are an obstruct to C*-simplicity, since the quotient map of groups $G \to G/N$ extends to a *-homomorphisms $\mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}^*_{\mathrm{red}}(G/N)$. We refer to [5, Appendix G] for basic properties of amenable groups. Notably, the amenable radical of G is its maximal normal amenable subgroup. For about 30 years combinatorial methods stayed at the heart of developments in C*-simplicity, trying to address the following main problem.

Problem 1. — *Clarify the relation between the following three statements for a discrete group G.*

- *G is* C*-*simple.*
- *G has the unique trace property.*
- *The amenable radical of G is trivial.*

Here a discrete group G is said to have the unique trace property if $\mathrm{C}^*_{\mathrm{red}}(G)$ admits a unique tracial state, that is a unique linear functional $\tau : \mathrm{C}^*_{\mathrm{red}}(G) \to \mathbb{C}$ such that $\tau(x^*x) \geq 0$ and $\tau(xy) = \tau(yx)$ for all $x, y \in \mathrm{C}^*_{\mathrm{red}}(G)$. While it was clear that every C*-simple group and every group with the unique trace property must have a trivial amenable radical, it was not even known whether every C*-simple group necessarily has the unique trace property, or even any idea of a proof that C*-simplicity or the unique trace property would have concrete implications on the structure of G.

The major breakthrough in the field of C*-simplicity was obtained in the combination of work by Kalantar–Kennedy in [37] and by Breuillard–Kalantar–Kennedy–Ozawa in [6]. At the heart of this development lies the following characterisation of C*-simplicity in terms of topological dynamics.

Theorem 2 ([37, Theorem 1.5] and [6, Theorem 1.1]). — *A discrete group G is* C*-*simple if and only if its action on the Furstenberg boundary is topologically free.*

The Furstenberg boundary is a compact G-space introduced in [23] and featuring in a different disguise in [30, Remark 3]. It is introduced in Definition 20. With this characterisation at hand, Breuillard–Kalantar–Kennedy–Ozawa solved virtually all open questions on C*-simplicity in [6]. In particular, the unique trace property could be definitively related to the structure of the group G by the following theorem.

Theorem 3 ([6, Theorem 1.3]). — *A discrete group has the unique trace property if and only if its amenable radical is trivial. In particular, every* C*-*simple group has the unique trace property.*

This result, together with Le Boudec's examples of groups with trivial amenable radical that are not C*-simple [42], clarified all general relations in Problem 1. However, in many classes of groups, an equivalence between these three statements can be proven, and it is another major contribution of [6] to prove easily applicable sufficient criteria for C*-simplicity, based on the notion of normalish subgroups: a subgroup $H \leq G$ is normalish if for every $g \in G$ the set $H \cap gHg^{-1}$ is infinite.

Theorem 4 ([6, Theorem 6.2]). — *A discrete group with no non-trivial finite normal subgroups and no amenable normalish subgroups is* C*-*simple.*

The present document focuses on the following tasks.

– In Section 1, we report on the notions in operator systems that led in [37] to the discovery of the connection between Furstenberg boundary and C*-simplicity.
– In Sections 2, 3 and 4 we report on the main achievements of [6] and provide a new proof for Theorem 2 which does not make any use of operator algebraic notions.
– In Section 5 we report on how these results were used by Kennedy and Haagerup in [39, 27] to relate back to original ideas of Powers.
– In Section 6 we summarise contemporary research related to C*-simplicity.

Given de la Harpe's exhaustive 2007 survey on C*-simplicity [13], we refrain from a more detailed presentation of developments in C*-simplicity before 2014, the year when the first versions of [37, 6] were published on arXiv. We provide a short list of references of articles on C*-simplicity between 2007 and 2014. Most of these articles addressed Problem 1 for particular classes of groups, proving equivalence between all three mentioned properties in the respective class. Linear groups were considered in [50], 3-manifold groups in [14], certain amalgamated free products in [34], convergence groups in [44] and acylindrically hyperbolic groups in [11]. It has to be pointed out that [50], following a series of revisions on arXiv, was never published, however the article's results are recovered as [6, Theorem 1.6] with an independent proof. A result in another direction can be found in [46], which provided the first examples of C*-simple groups without free subgroups, thereby showing the limits of general combinatorial ideas.

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1. BOUNDARIES

At the root of the breakthrough in C*-simplicity taking place since 2014 lies the discovery of connections between C*-algebras/representation theory on the one side and topological dynamics on the other side. While similar connections in a measurable setting, that is between von Neumann algebras and Poisson boundaries, were already successfully exploited in the past (e.g. in [10]), a major impact on C*-algebraic problems could only be observed after Kalantar–Kennedy [37] linked C*-simplicity with topological dynamics on two classical boundaries which were identified with each other: the Hamana boundary (Definition 15) and the Furstenberg boundary (Definition 20). Interestingly, this identification had already been stated by Hamana in [30, Remark 4], however the profound impact on our understanding of C*-simplicity had its advent in [37]. In this section we provide an exposition of both these boundaries, emphasising how their major properties —injectivity and essentiality on the side of the Hamana boundary, and universality and strong proximality on the side of the Furstenberg boundary— are related to each other.

1.1. The Hamana boundary

The Hamana boundary appears as a special instance of a more general theory of injective envelopes of operator algebras developed by Hamana in the 70s in [31, 32], in analogy with the theory of injective hulls of modules [18].

Although later accounts of C*-simplicity, notably [6], try to avoid the notion of operator systems and we provide an operator algebra free proof of the characterisation of C*-simplicity in Section 2, the setting of operator systems was crucial for the initial discovery in [37] that C*-simplicity and topological dynamics are closely related. The past teaches us that operator algebras are often essential to discover new theories that later can be formulated in more elementary language. In view of this lesson and in the hope of further developments, an account on C*-simplicity would not be complete without a discussion of the ideas from operator systems underlying the discovery of [37].

Let us begin by recalling some basic facts on operator systems which are necessary in what follows. On a historical note, operator systems were first used in [2] and obtained their name in [9, p. 157]. Note that every unital C*-algebra (when represented on a Hilbert space) is an operator system in the sense of the following definition.

Definition 5. — *An* operator system *is a unital, self-adjoint subspace of* $\mathcal{B}(H)$ *for some Hilbert space* H*.*

Recall that an operator in $x \in \mathcal{B}(H)$ is called positive if it self-adjoint and its spectrum lies in the positive half-line. Equivalently, $x = y^*y$ for some $y \in \mathcal{B}(H)$. An operator system $V \subset \mathcal{B}(H)$ is generated by the cone of its positive elements V^+, since $v = v^* \in V$ implies $v + \|v\|1 \in V^+$. Thus V inherits an order structure from positivity in $\mathcal{B}(H)$. The algebraic tensor products $\mathrm{M}_n(\mathbb{C}) \otimes V \subset \mathrm{M}_n(\mathbb{C}) \otimes \mathcal{B}(H)$ similarly inherits an order structure from positivity. In fact, operator systems admit an intrinsic characterisation in terms of these order structures, however here we only use it to define morphisms between operator systems.

Definition 6. — *A linear map* $\varphi : V \to W$ *between operator systems is called* completely positive*, if* $\varphi_n : \mathrm{M}_n(\mathbb{C}) \otimes V \to \mathrm{M}_n(\mathbb{C}) \otimes W$ *is positive for all* $n \in \mathbb{N}_{\geq 1}$*, that is the image of every positive element under* φ_n *is positive.*

Given a discrete group G, a G-operator system is an operator system V with an action of G by unital completely positive maps, which automatically are complete order isomorphisms. An equivariant unital completely positive map between G-operator systems is called G-unital completely positive map.

In analogy to the theory of C*-algebras, a state on an operator system V is a unital positive linear functional of norm one. The set of all states on V is denoted by $\mathcal{S}(V)$. It is a classical theorem of Stinespring that every state is a unital completely positive map [54]. Considering $\mathbb{C} = \mathrm{C}(\{pt\})$, this generalises to the following important correspondence between G-unital completely positive maps into commutative C*-algebras and continuous maps into the state space of an operator system. This observation provides the main connection between operator system theory and topological dynamics.

Proposition 7. — *Let V be a G-operator system and X a compact G-space. Composing a G-unital completely positive map $\Phi : V \to \mathrm{C}(X)$ with the evaluation maps $ev_x : \mathrm{C}(X) \to \mathbb{C}$, $ev_x(f) = f(x)$, we obtain a bijection between*

- *G-unital completely positive maps $\Phi : V \to \mathrm{C}(X)$.*
- *G-equivariant maps $X \to \mathcal{S}(V)$.*

Let us fix the relevant notion of injectivity, which is equivalent to the categorical definition.

Definition 8. — *A G-operator system V is called* injective *if for every inclusion of G-operator systems $X \subset Y$ and any G-unital completely positive map $X \to V$ there is an extension to a G-unital completely positive map $Y \to V$.*

In view of the previous discussion on states, it is clear that the Hahn–Banach theorem implies injectivity of the operator system $\mathbb{C}$. Further, Arveson's Extension Theorem [2, Theorem 1.2.3] says that the operator system $\mathcal{B}(H)$ are injective for arbitrary Hilbert spaces H.

Recall that an the injective hull of an R-module M is a monomorphism $M \hookrightarrow E$ into an injective R-module E, which is essential in the sense that any non-trivial submodule of E intersects M non-trivially. The following definition of Hamana provides the correct analogues of these properties for operator systems. We make use of the notion of complete isometries between operator systems, that is linear maps $\varphi : V \to W$ for which all amplifications $\varphi_n : \mathrm{M}_n(\mathbb{C}) \otimes V \to \mathrm{M}_n(\mathbb{C}) \otimes W$ are isometric. Completely isometric unital completely positive maps play the role of monomorphisms between operator systems.

Definition 9 ([32, Definitions 2.3 and 2.5]). — *Let $V \subset W$ be an inclusion of G-operator systems.*

- *The inclusion $V \subset W$ is called* essential*, if every G-unital completely positive map $\Phi : W \to X$ that is completely isometric on V must be completely isometric on W.*
- *The inclusion $V \subset W$ is an* injective envelope *of V if it is essential and W is injective. We denote it by $V \subset \mathrm{I}_G(V)$.*

From this definition of injective envelope, its uniqueness (up to isomorphism) is not immediately clear. It rather follows from the next rigidity property.

Definition 10 ([32, Definition 2.4]). — *An inclusion of G-operator systems $V \subset W$ is* rigid *if every G-unital completely positive endomorphism of W that restricts to the identity on V must be the identity on W.*

Hamana noted that for injective extensions of operator systems essentiality and rigidity are equivalent.

Lemma 11 ([32, Lemma 3.7]). — *Let $V \subset W$ be an inclusion of G-operator systems and assume that W is injective. Then $V \subset W$ is essential if and only if it is rigid.*

For discrete groups as they are considered in this exposition, the work of [32] extends from the setting of operator systems to G-operator systems, showing the following theorem.

Theorem 12 ([32, Theorem 4.1]). — *Every G-operator system admits an injective envelope.*

As Hamana states [32, p. 775], his work allows to show the next analogue of the Eckmann–Schopf's characterisation of injective hulls of modules [18, Statement 4.2]. This result provides motivation for the definition of an injective envelope. Since a proof is not spelled out in [32], but very short when following the idea of Eckmann–Schopf, we include it for the reader's convenience.

Proposition 13. — *Let $V \subset W$ be an inclusion of G-operator systems. The following properties are equivalent:*

- *$V \subset W$ is the injective envelope.*
- *W is a maximal essential extension of V.*
- *W is minimal injective over V.*

Proof. — Since essentiality is a transitive property, it suffices to prove that a G-operator system W is injective if and only if it is maximal essential over itself.

Assume that W is injective and let $W \subset X$ be an essential extension. By injectivity there is a G-unital completely positive map $\Phi : X \to W$ such that $\Phi|_W = \mathrm{id}_W$. So essentiality implies that Φ is injective, which shows $W = X$. Assuming conversely that W is maximal essential over itself, it must be equal to its injective envelope and hence injective. □

Given a discrete group G, a G-C*-algebra is a C*-algebra with an action of G by *-automorphisms. Moving towards the definition of the Hamana boundary of a group, we discuss the so called Choi–Effros product which can be used to define a G-C*-algebra structure on an injective envelope of a G-C*-algebra. In fact, the proof of [9, Theorem 3.1] is based on the existence of a projection $\mathcal{B}(H) \to V$ onto an operator system. Inspection of the proof shows independence of the concrete choice of $\mathcal{B}(H)$, which may be replaced by any C*-algebra. As a consequence, since every G-operator system can be embedded into a G-C*-algebra such as $\ell^\infty(G, \mathcal{B}(H))$ (cf. the remarks following Definition 8), the work of Choi–Effros shows the following statement.

Theorem 14 ([9, Theorem 3.1]). — *Every injective G-operator system V carries a product turning it into a G-C^*-algebra whose underlying operator system is V. This product is unique up to G-unital completely positive automorphisms of V. If V can be included into an abelian G-C^*-algebra, then this product on V is commutative.*

Applying this result to the special case of the trivial G-operator system $\mathbb{C}$, we obtain the Hamana boundary.

Definition 15. — *The* Hamana boundary *of a group G is the compact G-space $\partial_{\mathrm{H}} G = \mathrm{spec}(\mathrm{I}_G(\mathbb{C}))$, the injective envelope $\mathrm{I}_G(\mathbb{C})$ being considered as an abelian G-C^*-algebra.*

We are next going to provide a dynamical characterisation of essential extensions. This together with Proposition 23 makes it clear how essential extensions of G-operator system and strong proximality of compact G-spaces are related to each other.

A convex G-space is a convex G-invariant subset of a locally convex vector space with G-action by topological vector space isomorphisms. We call a compact convex G-space C irreducible if every closed convex G-subset of it is either empty or equals C. Further, observe that given two operator systems V, W a unital completely positive map $\varphi : V \to W$ is isometric if and only if $\varphi^* : \mathcal{S}(W) \to \mathcal{S}(V)$ is surjective. This is because of compactness of the state space combined with the fact that functionals are norming.

Proposition 16. — *Let $V \subset W$ be an inclusion of G-operator systems such that $G \curvearrowright \mathcal{S}(V)$ is irreducible. Then the following statements are equivalent.*

- *W is an essential extension of V*
- *$G \curvearrowright \mathcal{S}(W)$ is irreducible.*

Proof. — Assume that $V \subset W$ is essential. Let $\varphi \in \mathcal{S}(W)$ and denote by $\mathrm{P}_\varphi : W \to \ell^\infty(G)$ the Poisson map associated with φ, which satisfies $\mathrm{P}_\varphi(w)(g) \overset{def}{=} (g\varphi)(w)$. Since P_φ composed with any evaluation map on $\ell^\infty(G)$ is positive, it follows that P_φ is completely positive. Since $G \curvearrowright \mathcal{S}(V)$ is irreducible, it follows that for $v \in V$

$$\|\mathrm{P}_\varphi(v)\| = \sup_{g \in G} |(g\varphi)(v)| = \sup_{\psi \in \mathcal{S}(V)} |\psi(v)| = \|v\|.$$

Since $\ell^\infty(G)$ is abelian, this shows that $(P_\varphi)|_V$ is completely isometric. Because $V \subset W$ is essential, it follows that P_φ is completely isometric, so that $\mathrm{P}_\varphi^* : \mathcal{S}(\ell^\infty(G)) \to \mathcal{S}(W)$ is surjective. Since $\mathcal{S}(\ell^\infty(G))$ is the G-convex closure of ev_e, and $\mathrm{P}_\varphi(ev_e) = \varphi$, it follows that the G-convex closure of φ is $\mathcal{S}(W)$. Since φ was arbitrary, this shows irreducibility of $G \curvearrowright \mathcal{S}(W)$.

Assume now that $G \curvearrowright \mathcal{S}(W)$ is irreducible. In order to show that $V \subset W$ is essential, let $\Phi : W \to X$ be a G-unital completely positive map such that $\Phi|_V$ is completely isometric. Pick any $\varphi \in \mathcal{S}(X)$ and note that the G-convex closure of $\varphi \circ \Phi$ equals $\mathcal{S}(W)$ by irreducibility. Denote by $C \subset \mathcal{S}(X)$ the G-convex closure of φ and let $w \in W$. Then

$$\|w\| = \sup_{\psi \in \mathcal{S}(W)} |\psi(w)| = \sup_{\psi \in C}(\psi \circ \Phi(w)) \leq \|\Phi(w)\|.$$

It follows that Φ is isometric and hence completely isometric since it is completely positive. □

Applying the previous proposition to the injective envelope of the trivial G-operator system $\mathbb{C}$, we obtain an important statement about the Hamana boundary, which is to be seen in light of Proposition 23. In dynamical terms, the Hamana boundary is a G-boundary.

Corollary 17. — *For every group G the action $G \curvearrowright \mathcal{P}(\partial_{\mathrm{H}} G)$ is irreducible.*

Point stabilisers of the Hamana boundary are amenable.

Proposition 18 ([22, Proposition 7]). — *Let G be a group. For every $x \in \partial_{\mathrm{H}} G$, the fix group $G_x \leq G$ is amenable. Further, the kernel of $G \curvearrowright \partial_{\mathrm{H}} G$ is the amenable radical* $\mathrm{R}(G)$.

Proof. — Let $x \in \partial_{\mathrm{H}} G$. Since $\mathrm{C}(\partial_{\mathrm{H}} G)$ is an injective G-operator system, there is a G-unital completely positive projection $\ell^\infty(G) \to \mathrm{C}(\partial_{\mathrm{H}} G)$. Composing this map with the G_x-invariant evaluation map $ev_x : \mathrm{C}(\partial_{\mathrm{H}} G) \to \mathbb{C}$, we obtain a G_x-invariant state on $\ell^\infty(G_x) \subset \ell^\infty(G)$. This implies that the kernel of $G \curvearrowright \partial_{\mathrm{H}} G$ is amenable, so it is contained in the amenable radical.

For the reverse inclusion, let $\mu \in \mathcal{P}(\partial_{\mathrm{H}} G)$ be some probability measure fixed by the amenable radical of $\mathrm{R}(G)$. For every $g \in G$, normality of $\mathrm{R}(G)$ implies that $g\mu$ is $\mathrm{R}(G)$-fixed. Since $G \curvearrowright \mathcal{P}(\partial_{\mathrm{H}} G)$ is irreducible by Corollary 17, it follows that $\mathrm{R}(G)$ fixes every probability measure on $\partial_{\mathrm{H}} G$ and hence it lies in the kernel of $G \curvearrowright \partial_{\mathrm{H}} G$. □

A final important property of the Hamana boundary is that it is extremally disconnected in the sense that the closure of any of its open subsets is clopen. This property can be directly obtained from properties of operator systems.

Proposition 19. — *The Hamana boundary is extremally disconnected.*

Proof. — First observe that every injective G-operator system V is injective as an operator system without the G-action. This is because the G-operator system $\ell^\infty(G, \mathcal{B}(H))$ is injective as an operator system and any G-embedding $V \subset \ell^\infty(G, \mathcal{B}(H))$ admits a projection onto V by G-injectivity. Next, note that injectivity of the operator system V provides us with an embedding $V \subset \mathcal{B}(H)$ admitting a projection. It follows that V is monotone complete in the sense that every bounded set of self-adjoint elements in V has a least upper bound. It is well known that an abelian C^*-algebra is monotone complete if and only if its spectrum is extremally disconnected (see for example [26, Folk Theorem, p. 485] or [52, Theorem 4.3]). Specialising to $V = \mathrm{C}(\partial_{\mathrm{H}} G)$, we see that the Hamana boundary is extremally disconnected. □

1.2. Furstenberg boundary

The Furstenberg boundary is of dynamical origin and was introduced in [23] in order to study boundary theory of harmonic functions on connected Lie groups and develop a strategy for proving superrigidty of their lattices prior to work of Margulis. In [24], Glasner provided an excellent account of the Furstenberg boundary, and Ozawa prepared instructive lecture notes addressing the topic [48]. Yet another exposition of the Furstenberg boundary seems hence redundant, so we focus our discussion on making its relation to the Hamana boundary as transparent as possible.

In the remainder of this piece, we will identify a compact space X with a subset of the convex set of Borel probability measures $\mathcal{P}(X)$ identifying a point with its Dirac measure: $x \mapsto \delta_x$.

Definition 20 ([23, Definition 4.1]). — *Let X be a compact G-space.*

- *X is* strongly proximal *if for every probability measure $\mu \in \mathcal{P}(X)$ there is a net of group elements $(g_i)_i$ in G and some $x \in X$ such that $g_i\mu \to \delta_x$ in the weak-* topology.*
- *X is a G*-boundary *if it is minimal and strongly proximal.*
- *X is a* Furstenberg boundary *of G if it is a G-boundary and for every other G-boundary Y there is a G-equivariant continuous map $X \to Y$.*

The existence of the Furstenberg boundary follows from a product argument involving representatives of all boundaries. It is proved in [23, p. 199], [24, p. 32] and [48, p. 2].

The next proposition collects different aspects of rigidity visible in the dynamical setup. It is immediate to conclude uniqueness of the Furstenberg boundary up to unique isomorphisms from it.

Proposition 21 ([23, Proposition 4.2]). — *Let X be a strongly proximal G-space and Y be some compact G-space.*

- *The image of every G-equivariant map $Y \to \mathcal{P}(X)$ intersects X.*
- *If Y is minimal, the image of any G-equivariant map $Y \to \mathcal{P}(X)$ is contained in X and there is at most one G-equivariant map $Y \to X$.*
- *If X is a boundary, the image of any G-equivariant map $Y \to \mathcal{P}(X)$ contains X.*

Proof. — Consider a G-equivariant map $\varphi : Y \to \mathcal{P}(X)$ and pick $y_0 \in Y$. Since X is strongly proximal, there is $x \in X$ and a net $(g_i)_i$ in G such that $g_i\varphi(y_0) \to \delta_x$. Potentially passing to a subnet of $(g_i)_i$, compactness of Y allows to assume that $g_iy_0 \to y$ for some $y \in Y$. Continuity and G-equivariance then show $\varphi(y) = \delta_x$. So there is some $y \in Y$ such that $\varphi(y) \in X$.

If Y is minimal we further conclude that

$$\varphi(Y) = \varphi(\overline{Gy}) = \overline{G\delta_x} \subset X.$$

Moreover, if $\varphi_1, \varphi_2 : Y \to X$ are two G-equivariant continuous maps, then the map $y \mapsto \frac{1}{2}(\delta_{\varphi_1(y)} + \delta_{\varphi_2(y)}) \in \mathcal{P}(X)$ must take values in X. So $\varphi_1 = \varphi_2$.

If X is a boundary, we find that $\varphi(Y) \supset \overline{G\delta_x} = X$. □

Translating the previous proposition into the language of operator systems, we obtain rigidity in the sense of Definition 10 and essentiality in the sense of Definition 9. Recall at this point that rigidity and essentiality are not equivalent in general —cf. Hamana's Lemma 11.

COROLLARY 22. — *The extension of G-operator systems $\mathbb{C} \subset \mathrm{C}(\partial_{\mathrm{F}} G)$ is rigid and essential.*

Proof. — Proposition 7 establishes a correspondence between G-unital completely positive map $\mathrm{C}(\partial_{\mathrm{F}} G) \to \mathrm{C}(Y)$ and G-equivariant maps $Y \to \mathcal{P}(\partial_{\mathrm{F}} G)$. In particular, Proposition 21 implies that every G-unital completely positive endomorphism of $\mathrm{C}(\partial_{\mathrm{F}} G)$ equals the identity. So $\mathbb{C} \subset \mathrm{C}(\partial_{\mathrm{F}} G)$ is rigid.

In order to show essentiality of $\mathbb{C} \subset \mathrm{C}(\partial_{\mathrm{F}} G)$, consider an arbitrary G-unital completely positive map into a G-operator system $\mathrm{C}(\partial_{\mathrm{F}} G) \to V$. Composition with states of V provides us with a G-equivariant affine map $\mathcal{S}(V) \to \mathcal{P}(\partial_{\mathrm{F}} G)$. Since $\partial_{\mathrm{F}} G$ is a boundary, Proposition 21 says that its image contains $\partial_{\mathrm{F}} G$. Since it is affine, we conclude its surjectivity, so that $\mathrm{C}(\partial_{\mathrm{F}} G) \to V$ is isometric, and hence completely isometric since it is unital completely positive. This shows essentiality of $\mathbb{C} \subset \mathrm{C}(\partial_{\mathrm{F}} G)$. □

The next proposition puts the notion of G-boundaries into the context of operator systems, when combined with Proposition 16.

PROPOSITION 23 ([24, Section III.2]). — *A compact G-space X is a boundary if and only if the action $G \curvearrowright \mathcal{P}(X)$ is irreducible.*

Proof. — Assume that X is a G-boundary and let $\mu \in \mathcal{P}(X)$. By strong proximality, $\overline{G\mu} \cap X \neq \emptyset$ so that minimality implies $\overline{G\mu} \supset X$. The Krein–Milman theorem implies

$$\overline{\mathrm{conv}}\, G\mu \supset \overline{\mathrm{conv}}\, X = \mathcal{P}(X),$$

which proves that $G \curvearrowright \mathcal{P}(X)$ is irreducible.

Assume now that $G \curvearrowright \mathcal{P}(X)$ is irreducible. Let $\mu \in \mathcal{P}(X)$. Since $\overline{\mathrm{conv}}\, G\mu = \mathcal{P}(X)$ and X is the set of extreme points in $\mathcal{P}(X)$, Milman's converse to the Krein–Milman theorem (see e.g. [47, Proposition 5.26]) implies that $X \subset \overline{G\mu}$. Since μ was arbitrary, it follows that $G \curvearrowright X$ is minimal and strongly proximal. □

One can now establish that the Hamana and the Furstenberg boundaries are the same. This is formulated in the next theorem, whose statement can be already found without proof as Remark 4 in [30].

THEOREM 24 ([30, Remark 4] and [37, Theorem 3.11]). — *There is a unique isomorphism of compact G-spaces $\partial_{\mathrm{H}} G \cong \partial_{\mathrm{F}} G$.*

Proof. — Combining the characterisation of boundaries from Propositions 23 with the statement of Proposition 16, we see that the Hamana boundary $\partial_{\mathrm{H}}G$ is a boundary in the sense of Definition 20. Hence, there is a G-equivariant map $\partial_{\mathrm{F}}G \to \partial_{\mathrm{H}}G$, which dualises to a *-homomorphism $\mathrm{C}(\partial_{\mathrm{H}}G) \to \mathrm{C}(\partial_{\mathrm{F}}G)$. In the other direction, injectivity of $\mathrm{C}(\partial_{\mathrm{H}}G)$ applied to the inclusion $\mathbb{C} \subset \mathrm{C}(\partial_{\mathrm{F}}G)$ and the map $\mathbb{C} \hookrightarrow \mathrm{C}(\partial_{\mathrm{H}}G)$ provides a G-unital completely positive map $\mathrm{C}(\partial_{\mathrm{F}}G) \to \mathrm{C}(\partial_{\mathrm{H}}G)$. The latter translates by Proposition 7 to a G-equivariant map $\partial_{\mathrm{H}}G \to \mathcal{P}(\partial_{\mathrm{F}}G)$, whose image lies in $\partial_{\mathrm{F}}G$ by Proposition 21. This means that $\mathrm{C}(\partial_{\mathrm{F}}G) \to \mathrm{C}(\partial_{\mathrm{H}}G)$ is a *-homomorphism. Applying rigidity of $\mathrm{C}(\partial_{\mathrm{H}}G)$ (Proposition 11) and $\mathrm{C}(\partial_{\mathrm{F}}G)$ (Corollary 22), respectively, we conclude that the constructed *-homomorphisms are inverses of each other. Uniqueness of the induced isomorphism $\partial_{\mathrm{H}}G \cong \partial_{\mathrm{F}}G$ follows from Proposition 21. □

The following proposition provides a tool to produce many more maps from G-operator systems than injectivity of G-operator systems alone would allow. This is one of the most useful observations made in the dynamical setting.

Proposition 25 (Furstenberg, cf. [24, Chapter III, Theorem 2.3])
Let $G \curvearrowright C$ be an irreducible action on a compact convex set and denote by $\overline{\mathrm{ex}}(C)$ the closure of the extremal points of C. Then $\overline{\mathrm{ex}}(C)$ is a G-boundary.

Although we showed in Theorem 24 that the Furstenberg boundary is isomorphic with the Hamana boundary, one statement that does not immediately become clear is why $\mathrm{C}(\partial_{\mathrm{F}}G)$ is an injective G-operator system. It is hence instructive to present the direct proof of this fact provided by Ozawa in Theorem 6 of [48].

Proposition 26. — $\mathrm{C}(\partial_{\mathrm{F}}G)$ *is a G-injective operator system.*

Proof. — Choosing any $\mu \in \mathcal{P}(\partial_{\mathrm{F}}G)$ we obtain the Poisson map $\mathrm{P}_\mu : \mathrm{C}(\partial_{\mathrm{F}}G) \to \ell^\infty(G)$ defined by $\mathrm{P}_\mu(f)(g) = \int_{\partial_{\mathrm{F}}G} f(gx)\mathrm{d}\mu(x)$, which is G-equivariant, unital and positive, hence G-unital completely positive, because $\ell^\infty(G)$ is abelian. Further, Proposition 25 allows us to find a G-boundary inside $\mathcal{S}(\ell^\infty(G))$ and hence a map $\partial_{\mathrm{F}}G \to \mathcal{S}(\ell^\infty(G))$, which by Proposition 7 corresponds to a G-unital completely positive map $\Phi : \ell^\infty(G) \to \mathrm{C}(\partial_{\mathrm{F}}G)$. By rigidity of the Furstenberg boundary (Proposition 22), we find $\Phi \circ \mathrm{P}_\mu = \mathrm{id}_{\mathrm{C}(\partial_{\mathrm{F}}G)}$. It follows that P_μ is isometric onto its image and $\mathrm{P}_\mu \circ \Phi$ is a projection from $\ell^\infty(G)$ onto this image. Since $\ell^\infty(G)$ is G-injective by the Hahn–Banach theorem, it follows that $\mathrm{C}(\partial_{\mathrm{F}}G)$ is injective too. □

2. DYNAMICAL CHARACTERISATION OF C*-SIMPLICITY

In this section we provide proofs for the main characterisation of C*-simplicity stated as [37, Theorem 1.5] by Kalantar–Kennedy and as [6, Theorem 3.1] by Breuillard–Kalantar–Kennedy–Ozawa. While [37] used an operator system perspective, and [6] provided a partially topological dynamical proof, the proof presented here establishes a direct link between topological dynamics and representation theory, without any operator algebraic techniques. We thank Matthew Kennedy for allowing us to present these results which are based on joint conversations.

Let us begin with the representation theoretic reformulation of C*-simplicity. Note that the following characterisation of weak containment in terms of approximation of matrix coefficients allows to make norm estimates in the same situations where C*-simplicity does. An excellent reference for weak containment is [5, Appendix F]. Recall also the left-regular representation $\lambda : G \to \mathcal{U}(\ell^2 G)$ satisfying $\lambda_g \delta_h = \delta_{gh}$ for all $g, h \in G$. Given a unitary representation π of G, we denote its Hilbert space by H_π.

Definition 27. — *Let G be a group and π, ρ unitary representations of G.*

- *We say that π is* weakly contained *in ρ and write $\pi \prec \rho$, if for every vector $\xi \in H_\pi$, every $\varepsilon > 0$ and every finite subset $F \subset G$ there are vectors $\eta_1, \ldots, \eta_n \in H_\rho$ such that*

$$\forall g \in F : \quad \Big|\langle \pi(g)\xi, \xi\rangle - \sum_{i=1}^{n} \langle \rho(g)\eta_i, \eta_i\rangle\Big| < \varepsilon.$$

- *We say that π and ρ are* weakly equivalent *and write $\pi \sim \rho$, if $\pi \prec \rho$ and $\rho \prec \pi$.*
- *We say that π is* weakly regular *if $\pi \prec \lambda$.*

Theorem 28 ([20, Theorem 1.2]). — *A group G is* C**-simple if and only if every weakly regular unitary representation of G is weakly equivalent to its regular representation.*

The main link between dynamics and representation theory is made by the Koopman representation. We will not need any theory about Koopman representations, but want to point out for the expert that since we study boundary action in the sense of Definition 20, we typically consider non-singular actions which are not preserving any probability measure. Recall that an action on a non-trivial σ-finite measure space $G \curvearrowright (X, \nu)$ is called non-singular if ν is a quasi-invariant, that is $g\nu \sim \nu$ for all $g \in G$.

Definition 29. — *Let (X, ν) be a non-singular G-space. Then the associated* Koopman representation *κ of G is the unique unitary representation on $\mathrm{L}^2(X, \nu)$, which satisfies the following formula for all $f \in \mathrm{C}(X)$, $x \in X$ and $g \in G$ and for chosen representatives of the Radon–Nikodym derivatives.*

$$\kappa(g)f(x) = f(g^{-1}x)\Big(\frac{\mathrm{d}g\mu}{\mathrm{d}\mu}\Big)^{1/2}(x).$$

The crucial ingredient for the approach presented here is the following property of G-boundaries singled out by Breuillard–Kalantar–Kennedy–Ozawa.

Lemma 30 ([6, Lemma 3.7]). — *Let G be a non-trivial group and X a G-boundary. Then for every non-empty open subset $U \subset X$ and $\varepsilon > 0$, there is a finite subset $F \subset G \setminus \{e\}$ such that for every probability measure μ on $\partial_{\mathrm{F}} G$, there is $t \in F$ satisfying $\mu(tU) > 1 - \varepsilon$.*

Recall that a compact G-space X is topological free, if the fixed points in X of every non-trivial group element in G are nowhere dense. Equivalently, for every finite subset $F \subset G$ and every non-empty open subset $U \subset X$ there is a non-empty open subset $V \subset U$ such that the sets gV for $g \in F$ are pairwise disjoint. The next theorem clarifies how topological freeness translates to a property of the Koopman representation of a boundary. Its proof essentially follows the same lines as [6, Proposition 3.5], although its statement looks different.

Theorem 31. — *Let X be a G-boundary and κ the Koopman representation associated to some quasi-invariant measure on X. Then the following statements are equivalent.*

1. *X is topologically free.*
2. *For any unitary representation of G satisfying $\pi \prec \kappa$, we have $\lambda \prec \pi$.*
3. *$\lambda \prec \kappa$.*

Proof. — It is clear that 2 implies 3. Let us first assume 1 and show 2. Let π be a unitary representation of G satisfying $\pi \prec \kappa$. We have to show that $\lambda \prec \pi$. Let $F \subset G$ be a finite subset and $\varepsilon > 0$. It suffices to find a vector $\xi \in H_\pi$ such that

$$\forall g \in F : \quad |\delta_{e,g} - \langle \pi(g)\xi, \xi\rangle\rangle| < \varepsilon.$$

Let $U \subset X$ be an open subset such that $gU \cap U = \emptyset$ for all $g \in F \setminus \{e\}$. Let $E \subset G$ be a finite subset such that for every probability measure μ on X there is some $t \in E$ such that $\mu(tU) > 1 - \varepsilon^2$ (cf. [6, Lemma 3.7]). Let $\xi_0 \in H_\pi$ be a unit vector. Denote by ν the quasi-invariant measure with respect to which κ is constructed. Since $\pi \prec \kappa$, there are $\eta_1, \ldots, \eta_n \in \mathrm{L}^2(X, \nu)$ such that

$$\forall g \in \bigcup_{t \in E} tFt^{-1} : \quad \Big|\langle \pi(g)\xi_0, \xi_0\rangle - \sum_{i=1}^{m} \langle \kappa(g)\eta_i, \eta_i\rangle\Big| < \varepsilon.$$

Without loss of generality, we may assume that $\sum_{i=1}^n \|\eta_i\|_2^2 = 1$. Then $\mathrm{d}\mu = \sum_{i=1}^m |\eta_i|^2 \mathrm{d}\nu$ defines a probability measure on X and there is $t \in E$ such that $\mu(tU) > 1 - \varepsilon^2$.

For all $g \in F \setminus \{e\}$ the fact that $gU \cap U = \emptyset$ implies

$$\langle \kappa(t)\kappa(g)\kappa(t)^* \mathbb{1}_{tU}\eta_i, \mathbb{1}_{tU}\eta_i\rangle = \langle \kappa(g)\mathbb{1}_U \kappa(t)^*\eta_i, \mathbb{1}_U \kappa(t)^*\eta_i\rangle = 0.$$

Further for $g \in F \setminus \{e\}$, the Cauchy–Schwarz inequality shows that

$$\Big|\sum_{i=1}^{n} \langle \kappa(t)\kappa(g)\kappa(t)^*\eta_i, \mathbb{1}_{tU^c}\eta_i\rangle\Big|^2 \leq \left(\sum_{i=1}^{n} \|\eta_i\|_2^2\right) \mu(tU^c) \leq \varepsilon^2,$$

and similarly

$$\Big|\sum_{i=1}^{n} \langle \kappa(t)\kappa(g)\kappa(t)^* \mathbb{1}_{tU^c}\eta_i, \mathbb{1}_{tU}\eta_i\rangle\Big|^2 \leq \mu(tU^c) \left(\sum_{i=1}^{n} \|\mathbb{1}_{tU}\eta_i\|_2^2\right) \leq \varepsilon^2.$$

We put $\xi = \pi(t)^*\xi_0$ and obtain for $g \in F \setminus \{e\}$

$$|\langle \pi(g)\xi, \xi\rangle| \leq \Big|\langle \pi(tgt^{-1})\xi_0, \xi_0\rangle - \sum_{i=1}^{n} \langle \kappa(tgt^{-1}\eta_i, \eta_i\rangle\Big| + \Big|\sum_{i=1}^{n} \langle \kappa(tgt^{-1})\eta_i, \eta_i\rangle\Big| \leq 3\varepsilon,$$

while $|\langle \pi(e)\xi, \xi\rangle| = \|\xi\|^2 = 1$ holds. This shows $\lambda \prec \pi$ and thus proves 2.

Let us now assume 3 and prove 1. In other words, assuming $\lambda \prec \kappa$, we will show that $G \curvearrowright X$ is topologically free. Take $g \in G$ that admits a non-trivial open subset $U \subset \mathrm{Fix}_X(g)$. We have to show that g is trivial. To this end let, $t_1, \dots, t_k \in G$ be elements such that for every probability measure $\mu \in \mathcal{P}(X)$ there is some $t \in \{t_1, \dots, t_k\}$ satisfying $\mu(tU) > 8/9$. Put $F = \{t_1 g t_1^{-1}, \dots, t_k g t_k^{-1}\} \cup \{e\}$ and let $\xi_1, \dots, \xi_n \in \mathrm{L}^2(X)$ be such that

$$\forall f \in F : \Big|\langle \delta_{e,f} - \sum_{i=1}^{n} \langle \kappa(f)\xi_i, \xi_i \rangle\Big| < \frac{1}{3}.$$

Without loss of generality we may assume that $\sum_{i=1}^{n} \|\xi_i\|_2^2 = 1$. Then $\mathrm{d}\mu = \sum_{i=1}^{n} |\xi_i|^2 \mathrm{d}\nu$ defines a probability measure $\mu \in \mathcal{P}(X)$, where ν denotes the quasi-invariant measure from which κ is constructed.

Let $t \in \{t_1, \dots, t_k\}$ be such that $\mu(tU) > 8/9$. Then the fact that $U \subset \mathrm{Fix}_X(g)$ implies

$$\begin{aligned}
\Big|\sum_{i=1}^{n} \langle \kappa(t)\kappa(g)\kappa(t)^* \xi_i, \xi_i \rangle\Big| &= \Big|\sum_{i=1}^{n} \langle \kappa(t)\kappa(g)\kappa(t)^* (\mathbb{1}_{tU} + \mathbb{1}_{tU^c}) \xi_i, \xi_i \rangle\Big| \\
&= \Big|\mu(tU) + \sum_{i=1}^{n} \langle \kappa(t)\kappa(g)\kappa(t)^* \mathbb{1}_{tU^c} \xi_i, \xi_i \rangle\Big| \\
&\geq \frac{8}{9} - \Big(\sum_{i=1}^{n} \|\kappa(g)\kappa(t)^* \mathbb{1}_{tU^c} \xi_i\|_2^2\Big)^{1/2} \Big(\sum_{i=1}^{n} \|\kappa(t)^* \xi_i\|_2^2\Big)^{1/2} \\
&\geq \frac{8}{9} - \frac{1}{3}.
\end{aligned}$$

Since $t^{-1} g t \in F$, this implies $t^{-1} g t = e$ and hence $g = e$. □

We can now provide a purely dynamical proof of [37, Theorem 1.5] and [6, Theorem 3.1], respectively. The formulation we adopt for the following theorem appeared already as [6, Proposition 7.6]. It is interesting that at no point in the proof do we have to refer to operator algebraic notions, and instead work directly with the characterisation of C*-simplicity provided by Theorem 28.

Theorem 32. — *Let G be a group and X a G-boundary with some amenable point stabiliser. Then G is* C*-*simple if and only if X is topologically free.*

Proof. — Assume that X is topologically free and denote by κ the Koopman representation of X with respect to some quasi-invariant measure. Then $\lambda \prec \kappa$ by Theorem 31 so that $\pi \prec \lambda$ implies $\pi \prec \kappa$. Another application of Theorem 31 shows that $\lambda \prec \pi$. This proves C*-simplicity of G.

Assume now that G is C*-simple and take $x \in X$ with amenable stabiliser. Since G_x is amenable, it follows that $\lambda_{G/G_x} \prec \lambda$ (see e.g. Theorem G.3.2., Theorem F.3.5 and Example E.1.8 (ii) in [5]). So C*-simplicity implies $\lambda \sim \lambda_{G/G_x} \prec \kappa$, which in turn implies topological freeness of X by Theorem 31. □

Pointing out the limits of the operator algebra free approach presented in this section, we are not aware of a proof for the following statement that does not use operator algebraic techniques.

THEOREM 33 ([6, Proposition 7.8]). — *Assume that G is a discrete group with a G-boundary X such that there is $x \in X$ with G_x C^*-simple. Then G is C^*-simple.*

3. THE UNIQUE TRACE PROPERTY

Recall that a group G has the unique trace property, if $\mathrm{C}^*_{\mathrm{red}}(G)$ has a unique tracial state. Note that the formula $\tau(u_g) = \delta_{g,e}$ defines at least one such tracial state. In this section we provide a proof that G has the unique trace property if and only if its amenable radical is trivial, following the presentation of [6]. Drawing the analogy with Section 2, we also point out a representation theoretic characterisation of the unique trace property and ask for a proof relating this to triviality of the amenable radical without passing through operator algebras.

THEOREM 34 ([6, Theorem 4.1]). — *Every tracial state on $\mathrm{C}^*_{\mathrm{red}}(G)$ factors through the natural conditional expectation $\mathrm{E} : \mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}^*_{\mathrm{red}}(\mathrm{R}(G))$, which satisfies $\mathrm{E}(u_g) = \mathbb{1}_R(g)u_g$ for all $g \in G$. In particular, the following statements are equivalent for a group G.*

- *G has the unique trace property.*
- *The amenable radical of G is trivial.*

Proof. — Assume that G has the unique trace property. Denote by R the amenable radical of G. Since R is amenable we obtain a well-defined *-homomorphism $\epsilon : \mathrm{C}^*_{\mathrm{red}}(R) \to \mathbb{C}$ satisfying $\epsilon(u_g) = 1$ for all $g \in R$. Note that ϵ is a trace on $\mathrm{C}^*_{\mathrm{red}}(R)$. Since R is normal, the composition $\epsilon \circ \mathrm{E}$ with the natural conditional expectation remains tracial. By the unique trace property we find $\tau = \epsilon \circ \mathrm{E}$ and hence $R = \{e\}$.

Assume now that the amenable radical of G is trivial. Let φ be some tracial state, which we can consider as a G-unital completely positive map $\varphi : \mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}(\partial_{\mathrm{F}} G)$. Denote by $\tilde{\varphi} : \mathrm{C}(\partial_{\mathrm{F}} G) \rtimes_{\mathrm{red}} G \to \mathrm{C}(\partial_{\mathrm{F}} G)$ some G-unital completely positive extension, which exists by G-injectivity of $\mathrm{C}(\partial_{\mathrm{F}} G)$ as verified in Proposition 26. Restricting $\tilde{\varphi}$ to $\mathrm{C}(\partial_{\mathrm{F}} G)$ we obtain a G-unital completely positive endomorphism which must equal the identity map by rigidity of $\mathrm{C}(\partial_{\mathrm{F}} G)$ shown in Corollary 22. The theory of multiplicative domains [7, Proposition 1.5.7] then shows that $\tilde{\varphi}$ is $\mathrm{C}(\partial_{\mathrm{F}} G)$-bimodular. So we obtain for arbitrary $f \in \mathrm{C}(\partial_{\mathrm{F}} G)$ and $g \in G$ that

$$\tau(u_g)f = \tilde{\varphi}(u_g)f = \tilde{\varphi}(u_g f) = \tilde{\varphi}(f(g^{-1}\cdot)u_g) = \tau(u_g)f(g^{-1}\cdot).$$

We can now invoke Furman's Proposition 18 saying that the kernel of the action $G \curvearrowright \partial_{\mathrm{F}} G$ is the amenable radical of G. So for $g \in G \setminus \mathrm{R}(G)$ there is some $x \in \partial_{\mathrm{F}} G$ satisfying

$gx \neq x$. Taking $f \in \mathrm{C}(\partial_{\mathrm{F}} G)$ some function such that $f(x) = 1$ and $f(g^{-1}x) = 0$, then $\tau(u_g) = \tau(u_g)f(x) = \tau(u_g)f(g^{-1}x) = 0$ follows. □

The following folklore representation theoretic characterisation of the unique trace property provides an analogue of Fell's Theorem 28. It would be interesting to find a proof of Theorem 34 that directly relates this characterisation to triviality of the amenable radical. Recall that two unitary representations π_1, π_2 of a group G are quasi-equivalent if there is a cardinal κ such that $\pi_1^{\oplus\kappa} \cong \pi_2^{\oplus\kappa}$. Further, we call a representation finite if it is generated by tracial vectors, that is vectors ξ whose matrix coefficient $\langle \cdot\xi, \xi\rangle$ defines a conjugation invariant function on G. Note that every finite dimensional representation is finite, but that the latter class is larger, containing the left-regular representation in particular.

In the proof of the next proposition we are going to employ the GNS-representation associated with a state on a C*-algebra, which is explained in [45, Sections 3.4 and 5.1]. It is noteworthy that the GNS-representation of the natural trace on $\mathrm{C}^*_{\mathrm{red}}(G)$ is unitary equivalent with the left-regular representation of a discrete group G.

Proposition 35. — *The following statements are equivalent for a group G.*

1. *G has the unique trace property.*
2. *Every finite, weakly regular unitary representation is quasi-equivalent to λ.*

Proof. — Assume that $\mathrm{C}^*_{\mathrm{red}}(G)$ has a unique trace. Let $\pi \prec \lambda$ be a finite unitary representation. We may assume without loss of generality that π is cyclic with tracial cyclic unit vector ξ. By linearity and continuity, the state $a \mapsto \langle a\xi, \xi\rangle$ on $\mathrm{C}^*_\pi(G)$ is tracial. Since $\pi \prec \lambda$, there is a *-homomorphism $\mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}^*_\pi(G)$ and we thus obtain a tracial state φ_ξ on $\mathrm{C}^*_{\mathrm{red}}(G)$. Then $\varphi_\xi = \tau$ by the unique trace property. Applying the GNS-theorem, we find that $\pi \cong \pi_{\varphi_\xi} = \pi_\tau \cong \lambda$.

Now assume the every finite weakly regular unitary representation is quasi-equivalent to λ. We first observe that the group von Neumann algebra of G is a factor and hence has a unique trace. Let φ be any tracial state on $\mathrm{C}^*_{\mathrm{red}}(G)$. Denote by π its GNS-representation, which is quasi-equivalent to λ by our assumption, so that $\pi(\mathrm{C}^*_{\mathrm{red}}(G))'' \cong \mathrm{L}G$ preserving the inclusion of $\mathrm{C}^*_{\mathrm{red}}(G)$. Since φ extends to a trace on the former von Neumann algebra, while the latter has a unique trace, it follows that $\varphi = \tau$. □

4. EXTENSIONS AND EXAMPLES OF C*-SIMPLE GROUPS

An important contribution of [6] was the systematic solution of virtually all open problems on C*-simplicity of discrete groups. We do not have anything to add to the elegant proofs presented in [6], which naturally make use of the dynamical characterisation of C*-simplicity provided by Theorem 32.

The first result to mention solved the longstanding open problem whether the extension of C*-simple groups remains C*-simple. The answer interestingly is yes. While the proof of [6, Theorem 1.4] constructs a topological free boundary action from the assumptions, we point out that stability of C*-simplicity under extensions is a corollary of Theorem 32.

THEOREM 36 ([6, Theorem 1.4]). — *Let G be a group and $N \trianglelefteq G$ a normal subgroup. Then G is* C*-*simple if and only if N and $\mathcal{Z}_G(N)$ are* C*-*simple. In particular,* C*-*simplicity is closed under extensions.*

Proof that C*-*simplicity is closed under extensions.* — Let G be a group with C*-simple normal subgroup $N \trianglelefteq G$ and C*-simple quotient $Q = G/N$. Consider the Furstenberg boundary $\partial_{\mathrm{F}}Q$ as a G-space. Since Q is C*-simple, its action on $\partial_{\mathrm{F}}Q$ is free by Theorem 32 and the fact that $\partial_{\mathrm{F}}Q$ is extremally disconnected (cf. [6, Proposition 2.4]). So $G \curvearrowright \partial_{\mathrm{F}}Q$ is a G-boundary whose point stabilisers are equal to N and hence C*-simple. An application of Theorem 33 finishes the proof. □

In another direction, [6] gave a series of criteria for groups to be C*-simple, recovering basically all previously known classes of examples. In view of Problem 1 about the relation between C*-simplicity and triviality of the amenable radical, we choose the following form to summarise these results and refer to [6] more details.

THEOREM 37 ([6, Theorems 1.5, 1.6 and 1.7]). — *A group G with trivial amenable radical is* C*-*simple if one of the following conditions holds.*

- *G is linear.*
- *Bounded cohomology of G does not vanish.*
- *There is at least one non-trivial ℓ^2-Betti number of G.*
- *G has at most countably many amenable subgroups.*

5. RECONNECTING TO ORIGINAL IDEAS. WORK OF HAAGERUP AND KENNEDY

5.1. The Dixmier property and Powers averaging

A unital C*-algebra A is said to have the *Dixmier property* if for any $a \in A$ the convex norm closure of $\{uau^* \mid u \in A \text{ unitary}\}$ intersects the centre of A non-trivially. This property was introduced in [16] in the context of von Neumann algebras and it is not difficult to show that a tracial, unital C*-algebra with trivial centre and the Dixmier property must be simple and has exactly one trace. Vice versa it was shown in [29, Corollaire] (see also [1] for a more recent generalisation) that a simple, unital C*-algebra with at most one trace has the Dixmier property.

For unital C*-algebras with trace the only scalar in $\overline{\mathrm{conv}}\{uau^* \mid u \in A \text{ unitary}\}$ can be the value of the trace on a. This allows us to rewrite the Dixmier property as follows:

a unital C*-algebra A with trace τ and trivial centre has the Dixmier property if and only if for every $a \in A$ there is a sequence of unitaries $(u_i)_{i\in\mathbb{N}}$ in $\mathcal{U}(A)$ such that

$$\left\|\frac{1}{n}\sum_{i=1}^{n} u_i a u_i^* - \tau(a)1\right\| \to 0.$$

With this reformulation in mind, we say that a group G satisfies the Powers averaging property [39, Definition 1.3] if for every element $a \in \mathrm{C}^*_{\mathrm{red}}(G)$ there is a sequence of group elements $(g_i)_{i\in\mathbb{N}}$ in G such that

$$\left\|\frac{1}{n}\sum_{i=1}^{n} u_{g_i} a u_{g_i}^* - \tau(a)1\right\| \to 0.$$

Using the original formulation of the Dixmier property, the last statement is equivalent to the convex norm closure of $\{u_g a u_g^* \mid g \in G\}$ intersecting the scalars of $\mathrm{C}^*_{\mathrm{red}}(G)$ non-trivially. Using this terminology, Powers' achievement was to single out a combinatorial way to ensure the Powers averaging property. After the results of [6], it appears natural to reconnect C*-simplicity to this original idea. Both Kennedy and Haagerup approached this problem independently, obtaining essentially equivalent but differently presented results. Note the resemblance with the proof of Theorem 34.

Theorem 38 ([39, Theorem 1.4] and [27, Theorem 5.3]). — *The following statements are equivalent for a group G with natural trace τ on $\mathrm{C}^*_{\mathrm{red}}(G)$.*

1. *G is C*-simple.*
2. *The unique G-unital completely positive map $\mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}(\partial_{\mathrm{F}} G)$ is $a \mapsto \tau(a)1$.*
3. *For every state $\varphi \in \mathrm{C}^*_{\mathrm{red}}(G)^*$ we have $\tau \in \overline{\mathrm{conv}}^{w^*}\{g\cdot\varphi \mid g \in G\}$.*
4. *G has Powers averaging property.*

Proof. — Assume that G is C*-simple and let $\Phi : \mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}(\partial_{\mathrm{F}} G)$ be any G-unital completely positive map. Since $\mathrm{C}(\partial_{\mathrm{F}} G)$ is G-injective by Proposition 26, it extends to a G-unital completely positive map $\Psi : \mathrm{C}(\partial_{\mathrm{F}} G) \rtimes_{\mathrm{r}} G \to \mathrm{C}(\partial_{\mathrm{F}} G)$. Restricting Ψ to $\mathrm{C}(\partial_{\mathrm{F}} G)$ and applying rigidity from Corollary 22, we find that $\Psi|_{\mathrm{C}(\partial_{\mathrm{F}} G)} = \mathrm{id}_{\mathrm{C}(\partial_{\mathrm{F}} G)}$. So the theory of multiplicative domains [7, Proposition 1.5.7] shows that Ψ is $\mathrm{C}(\partial_{\mathrm{F}} G)$-bimodular. As in the proof of Theorem 34, we conclude that $\Phi(u_g)f = \Phi(u_g)f(g^{-1}\cdot)$ for any $f \in \mathrm{C}(\partial_{\mathrm{F}} G)$. For every $x \in \partial_{\mathrm{F}} G \setminus \mathrm{Fix}(g)$, there is some function $f \in \mathrm{C}(\partial_{\mathrm{F}} G)$ such that $f(x) = 1$ and $f(g^{-1}x) = 0$. Hence

$$\Phi(u_g)(x) = \Phi(u_g)(x)f(x) = \Phi(u_g)(x)f(g^{-1}x) = 0.$$

Since G is C*-simple, Theorem 32 says that $G \curvearrowright \partial_{\mathrm{F}} G$ is topologically free, meaning that $\partial_{\mathrm{F}} G \setminus \mathrm{Fix}_{\partial_{\mathrm{F}} G}(g)$ is dense in $\partial_{\mathrm{F}} G$. So $\Phi(u_g) = 0 = \tau(u_g)$. This proves 2.

Assume 2 and let $\varphi \in \mathrm{C}^*_{\mathrm{red}}(G)^*$ be a state. Then $C = \overline{\mathrm{conv}}^{w^*}\{g\cdot\varphi \mid g \in G\}$ is a compact convex G-space, so that it contains a G-boundary $B \subset C$ by Proposition 25. Universality of the Furstenberg boundary provides us with a G-equivariant map $\partial_{\mathrm{F}} G \to B$, which then translates by Proposition 7 to a G-unital completely positive map $\Phi : \mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}(\partial_{\mathrm{F}} G)$. By 2 we have $\Psi(a) = \tau(a)1$ for all $a \in \mathrm{C}^*_{\mathrm{red}}(G)$, which translates to $B = \{\tau\}$, proving 3.

The implication 3 $\Longrightarrow$ 4 is an easy application of the Hahn–Banach separation theorem, while the implication 4 $\Longrightarrow$ 1 follows from the discussion about the Dixmier property. □

Note that the last result can be interpreted as saying that any non-trivial boundary action in the state space of the reduced group C*-algebra, already witnesses failure of C*-simplicity.

Let us mention the following characterisation of triviality of the amenable radical, highlighting the subtle difference to C*-simplicity.

Theorem 39 ([27, Theorem 5.2 (iv)]). — *A discrete group G has trivial amenable radical if and only if for every $h \in G$ we have that $\overline{\mathrm{conv}}\{u_g u_h u_g^* \mid g \in G\}$ intersects the scalars.*

5.2. The amenable radical

While [6] showed that a group with trivial amenable radical has the unique trace property, examples provided by Le Boudec in [42] showed that C*-simplicity cannot be concluded in general, as there are examples of groups with trivial amenable radical that are not C*-simple. It thus became natural to investigate to which extent the long hoped for equivalence between C*-simplicity and the triviality of the amenable radical could be saved. This was achieved by Kennedy in [39] replacing normal subgroups by the appropriate dynamical notion, so called uniformly recurrent subgroups [25] introduced by Glasner–Weiss.

Definition 40 ([25, Defintion 0.1]). — *Let G be a group.*

- *The* Chabauty space $\mathcal{S}(G)$ *is the compact G-space obtained from the set of all its subgroups equipped with the topology inherited from the power set 2^G and the action of G by conjugation.*
- *A minimal G-invariant closed subset of $\mathcal{S}(G)$ is called a* uniformly recurrent subgroup *of G.*
- *A uniformly recurrent subgroup of G is* amenable *if all its elements are amenable.*

The topology of the Chabauty space $\mathcal{S}(G)$ can be characterised by convergence of nets: a net of subgroups $(H_i)_{i \in I}$ converges to H, if for every $g \in G$ we have $g \in H_i$ ultimately if and only if $g \in H$.

Let us state an observation, providing a dynamical replacement of the amenable radical.

Lemma 41 (Cf. [39, Proposition 3.2 and Theorem 4.1]). — *For any discrete group G, the G-set $\{G_x \mid x \in \partial_{\mathrm{F}} G\} \subset \mathcal{S}(G)$ is an amenable uniformly recurrent subgroup of G.*

Proof. — Consider the map $\partial_{\mathrm{F}} G \to \mathcal{S}(G) : x \mapsto G_x$. It is G-equivariant and continuous since for every $g \in G$ the set $\mathrm{Fix}_X(g)$ is clopen, thanks to the fact that $\partial_{\mathrm{F}} G$ is extremally disconnected. The image of a minimal compact G-space under a continuous map has the same properties, so it follows that $\{G_x \mid x \in \partial_{\mathrm{F}} G\}$ is a uniformly recurrent subgroup. □

The uniformly recurrent subgroup defined in the previous statement is called the *Furstenberg uniformly recurrent subgroup* of G. It is the dynamical analogue of the amenable radical.

Theorem 42 ([39, Theorem 1.2]). — *For a group G the following statements are equivalent.*

- *G is C^*-simple.*
- *G has no non-trivial amenable uniform recurrent subgroups.*
- *The Furstenberg uniformly recurrent subgroup of G is trivial.*

If the amenable radical is the maximal amenable normal subgroup, one might wonder how the Furstenberg uniformly recurrent subgroup can be characterised in an algebraic way. In this direction, the proof of Theorem 6.2 in [6] and work of Kennedy [39, Section 5] can be interesting starting points.

6. DEVELOPMENTS FOLLOWING BREUILLARD–KALANTAR–KENNEDY–OZAWA

The presented work on C^*-simplicity gave rise to a wave of results on simplicity or more generally the ideal structure of C^*-algebras associated with groups. In this section we intend to give a concise overview of these developments pointing out current and future directions of research. Above all, it should be noted that the methods developed in [37] and [6] are of equal importance as the results.

C^*-simplicity of non-discrete groups:

Already in [12, page 232, Conjecture] speculations about the existence of non-discrete C^*-simple groups were raised and the problem to find one was expressed in [13, Question 5]. This question was made more precise in [8, Problem 8.1], asking to characterise C^*-simple groups "in terms of [their] Furstenberg boundary". While candidates for non-discrete C^*-simple groups considered in [12, Conjecture, page 232] are not C^*-simple due to the fact that every C^*-simple group must be totally disconnected [51, Theorem A], Suzuki provided in [55] some examples of non-discrete C^*-simple groups, whose construction is based on elementary considerations only using Powers' original work [49]. It is to be noted that it is currently unclear whether or not the examples of non-discrete C^*-simple groups provided in [51, Theorem B] are actually valid, owing to a gap in Lemma 5.1 pointed out by Suzuki. The natural generalisation of the unique trace property for non-discrete groups asks whether the reduced group C^*-algebra of a unimodular locally compact group admits the Plancherel trace as its unique (unbounded) trace up to scaling. Suzuki's examples do have this property. In another direction, [21] investigated which reduced group C^*-algebras admit bounded traces, and [40] solved this problem completely stating that $\mathrm{C}^*_{\mathrm{red}}(G)$ admits a tracial state if and only if the amenable radical of G is open.

Ideal structure of crossed products:

In [15], de la Harpe and Skandalis showed that if G is a Powers group, and A is a G-C*-algebra that does not admit any non-trivial G-invariant ideals, then the crossed product C*-algebra $A \rtimes_{\mathrm{red}} G$ is simple. Clearly the condition on A (called G-simplicity) is necessary. The condition on G can be weakend to C*-simplicity as [6, Theorem 7.1] shows. Methods developed in the setting of C*-simplicity, notably from [39], do not stop to apply here, and results on the ideal structure of crossed products by not-necessarily C*-simple groups are obtained in [38, 41]. See also [19, Section 6.3].

Characterisations of C*-simplicity through stationary traces:

In Section 5.2 it was shown that the gap between uniqueness of a trace and C*-simplicity can be overcome by a uniqueness result for G-unital completely positive maps into the Furstenberg boundary. In another direcrtion [33] investigated the possibility to relax the traciality condition and studied more generally stationary states on group C*-algebras to obtain another characterisation of C*-simplicity.

Twisted group C*-algebras:

Twisted group C*-algebras and twisted crossed products play an important role in the understanding of unitary representation theory, being linked through the notion of projective representation [3]. Although they appear in different disguise, the study of their simplicity even predates Powers' article [53]. Continuing work that had been done in between and possibly inspired by the recent success of C*-simplicity, [4] investigates simplicity and the unique trace property for twisted group C*-algebras. However, methods and ideas from [37, 6] could not yet be applied, leaving the possibility for further development.

Thompson's groups:

The question whether Thompson's group F is amenable or not, is very well-known in the group theory community. Surprisingly, it is possible to exhibit a sharp dichotomy between amenability and C*-simplicity in this context: after [28, Theorem 5.5] showed that C*-simplicity of Thompson's group T implies non-amenability of Thompson's group F, methods from [6] and [39] were applied to groups of homeomorphisms obtaining not only the converse to Haagerup–Olesen, but even the dichotomy that F is either amenable or C*-simple [43, Theorem 1.7].

Strong amenability:

The notion of strong amenability was introduced by Glasner in [24, Section II.3]. Being formulated in terms of proximal actions, it dynamically bears similarity to amenability characterised by triviality of the Furstenberg boundary. Seemingly independent from the developments on C*-simplicity, [36] characterised strongly amenable groups as FC-hypercentral groups (those groups that do not have any icc quotient) and [35] went on to solve the longstanding open problem to characterise finitely generated Choquet–Deny groups, which are by definition those finitely generated groups all of whose Poisson boundaries vanish. The power of dynamical methods and closeness to the operator algebraic setting are stunning similarities to the developments in C*-simplicity.

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