Von Neumann algebras and measured group theory

Lecture notes

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Introduction

I typed these notes during winter term 2015/16 for a lecture on "von Neumann algebras and measured group theory" given at the University of Münster, Germany. I intended to give a direct introduction to II_1 factors, circumventing any unnecessary complications that arise in the theory of non-finite von Neumann algebras. The field of II_1 factors developed rapidly during the last decade under the influence of Sorin Popa and Stefaan Vaes. This makes me feel that it is a good moment to introduce people working and studying in an environment mainly dominated by C^*-algebras to contemporary results and techniques in theory of II_1 factors. My course in Münster was aimed at an inhomogeneous audience, consisting of master students specialising in operator algebras as well as doctoral students and post doctoral researchers working with C^*-algebraic topics. Hopefully, these notes are a fruitful reference for both.

1 Von Neumann algebras, II_1 factors and Cartan subalgebras

Definition 1.0.1 (Von Neumann algebra)**.** Let H be a (complex) Hilbert space and denote by $\mathcal{B}(H)$ the *-algebra of bounded linear operators on H.

- The topology of pointwise convergence on $\mathcal{B}(H)$ is called the *strong operator topology* (SOT). We have $x_\lambda \to x$ in the SOT if and only if $x_\lambda \xi \to x\xi$ for all $\xi \in H$.
- The topology of pointwise weak convergence $\mathcal{B}(H)$ is called the *weak operator topology* (WOT). We have $x_\lambda \to x$ in the WOT if and only if $\langle x_\lambda \xi, \eta \rangle \to \langle x\xi, \eta \rangle$ for all $\xi, \eta \in H$.

A von Neumann algebra is a strongly closed unital *-subalgebra of $\mathcal{B}(H)$.

Remark 1.0.2. The Cauchy-Schwarz inequality says that $\langle T\xi, \eta \rangle \leq \|T\xi\| \|\eta\|$, which shows that SOT convergence of a net implies WOT convergence. Put differently, every weakly closed set is also strongly closed.

Example 1.0.3.

- $\mathcal{B}(H)$ itself is a von Neumann algebra.
- If $A \subset \mathcal{B}(H)$ is a unital C^*-algebra, then $\overline{A}^{\mathrm{SOT}}$ is a von Neumann algebra.
- If $S \subset \mathcal{B}(H)$ is any symmetric (i.e. $S^* = S$) subset, then $S' = \{x \in \mathcal{B}(H) \mid \forall y \in S : xy = yx\}$ is a von Neumann algebra. (Exercise!)

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- Let λ be the restricted Lebesgue measure on $[0,1]$. We can represent $L^\infty([0,1],\lambda)$ on $L^2([0,1],\lambda)$ by pointwise multiplication. One can show that $L^\infty([0,1],\lambda)' = L^\infty([0,1],\lambda)$. So $L^\infty([0,1],\lambda)$ is a von Neumann algebra.
- If Γ is a discrete group, then its left-regular representation $\Gamma \to \mathcal{U}(\ell^2(\Gamma))$ is defined by $\lambda_g \delta_h = \delta_{gh}$. We denote by $L(\Gamma) = \overline{\mathbb{C}\Gamma}^{\mathrm{SOT}}$ the group von Neumann algebra of Γ.

Definition 1.0.4 (Commutant)**.** Let $S \subset \mathcal{B}(H)$ be any subset. Then $S' = \{x \in \mathcal{B}(H) \mid \forall y \in S : xy = yx\}$ is called the *commutant of* S. More generally, if $M \subset \mathcal{B}(H)$ is a von Neumann algebra $S' \cap M$ is called the *relative commutant of* S *in* M.

The following proposition describes a basic link between a geometric property (invariance of a subspace) and an algebraic property invoking commutants. Despite its simple proof, it is the main ingredient of the Bicommutant Theorem's proof (See Theorem 1.0.6).

Proposition 1.0.5. *Let $A \subset \mathcal{B}(H)$ be a *-algebra, $K \leq H$ a closed Hilbert subspace and $p : H \to K$ the orthogonal projection onto K.*

- *If $p \in A$, then K is invariant under $A' \subset \mathcal{B}(H)$.*
- *If K is invariant under A, then $p \in A'$.*

Proof. Assume that $p \in A$ and let $x \in A'$. Then $xp = pxp$, meaning that $xK \subset K$. Vice versa, if we assume that K is invariant under A, then for every $x \in A$ we obtain $xp = pxp$. Since x is arbitrary and A is a *-algebra, we obtain also $px = pxp$ for all $x \in A$. This implies $xp = px$ for all $x \in A$, meaning that $p \in A'$. □

Theorem 1.0.6 (Bicommutant theorem)**.** *Let $M \subset \mathcal{B}(H)$ be a unital *-subalgebra. Then the following statements are equivalent.*

- *$M = M''$.*
- *M is closed in the WOT*
- *M is closed in the SOT*

Proof. We first show that every commutant is weakly closed. So let $S \subset \mathcal{B}(H)$ be some set and $T_\lambda \to T$ be some net in S' whose WOT limit lies in $\mathcal{B}(H)$. For arbitrary $\xi, \eta \in H$ and $s \in S$, we obtain

$$\begin{aligned}
\langle Ts\xi, \eta\rangle &= \lim_\lambda \langle T_\lambda s\xi, \eta\rangle \\
&= \lim_\lambda \langle sT_\lambda \xi, \eta\rangle \\
&= \lim_\lambda \langle T_\lambda \xi, s^*\eta\rangle \\
&= \langle T\xi, s^*\eta\rangle \\
&= \langle sT\xi, \eta\rangle .
\end{aligned}$$

Since $\xi, \eta \in H$ were arbitrary, this implies $Ts = sT$. So $T \in S'$.

By Remark 1.0.2 every weakly closed set is also strongly closed. So it suffices to show the following statement in order to complete the proof. If $M \subset \mathcal{B}(H)$ is a unital *-subalgebra, then $M'' = \overline{M}^{\mathrm{SOT}}$.

To this end, note that M'' is a strongly closed set containing M. So $M'' \supset \overline{M}^{\mathrm{SOT}}$. Now let $x \in M''$, $\varepsilon > 0$ and $\xi_1, \dots, \xi_n \in H$. We have to find $y \in M$ such that $\|(x - y)\xi_i\| < \varepsilon$ for all $i \in \{1, \dots, n\}$. Consider $\mathcal{B}(H)$ acting on $\bigoplus_{i=1}^n H$ diagonally and denote $\xi = \bigoplus_{i=1}^n \xi_i$. The subspace $K = \overline{M\xi} \subset \bigoplus_{i=1}^n H$ is invariant under the action of M. Hence, the orthogonal projection $p_K : \bigoplus_{i=1}^n H \to K$ is contained in the commutant $M' \cap \mathcal{B}(\bigoplus_{i=1}^n H)$. So $M'' \subset (M' \cap \mathcal{B}(\bigoplus_{i=1}^n H))' \cap \mathcal{B}(\bigoplus_{i=1}^n H)$ leaves K-invariant. We obtain that

$$x\xi = x1\xi \in xK \subset K .$$

By definition of K $(= \overline{M\xi})$ there is an element $y \in M$ such that $\|x\xi - y\xi\| < \varepsilon$. Now $\|x\xi - y\xi\|^2 = \sum_{i=1}^n \|(x - y)\xi_i\|^2$ implies that $\|(x - y)\xi_i\| < \varepsilon$ for all $i \in \{1, \dots, n\}$. This is what we had to show. □

Remark 1.0.7. Note that the last two items of Theorem 1.0.6 are equivalent by a more fundamental statement: every strongly continuous functional on $\mathcal{B}(H)$ is weakly continuous. This implies that a convex sets is strongly closed if and only if it is weakly closed. (See for example p.127, Theorem 4.2.6 of Murphy's book on C*-algebras.)

Remark 1.0.8. The previous theorem characterises von Neumann algebras by two conditions of a very different nature. One is algebraic, the other one is analytic. This should be considered as a hint that we just found a mathematically very rich and interesting structure.

Definition 1.0.9 (Separable von Neumann algebra)**.** A von Neumann algebra M is called separable, if it acts on a separable Hilbert space.

1.1 Normal positive functionals on von Neumann algebras

We already saw that there are at least two natural topologies on a von Neumann algebra. They are the most suitable to introduce von Neumann algebras. However, the right notion of continuity for maps on von Neumann algebras is given by yet another topology.

Definition 1.1.1 (σ-weak topology)**.** Let H be a complex Hilbert space. A net $(x_\lambda)_\lambda$ of operators in $\mathcal{B}(H)$ converges in the *σ-weak topology* to $x \in \mathcal{B}(H)$ if for all 2-summable sequences $(\xi_n)_n, (\eta_n)_n$ in H we have $\sum_n \langle x_\lambda \xi_n, \eta_n\rangle \to \sum_n \langle x\xi_n, \eta_n\rangle$.

There are several reasons why the σ-weak topology is natural. One fundamental theorem in the theory of von Neumann algebras identifies it with the weak-* topology induced by some Banach space duality $M = (M_*)^*$, where M_* is actually a uniquely determined Banach space. Here we are interested in characterising positive σ-weakly continuous functionals as those which resemble probability measures (and hence obey Σ-additivity).

Proposition 1.1.2. *Let $\varphi : M \to \mathbb{C}$ be a positive linear functional on a von Neumann algebra. Then φ is σ-weakly continuous if and only if it is* normal *in the following sense: for all bounded monotone sequences $x_1 \leq x_2 \leq \cdots \leq \lambda 1$ of self-adjoint elements in M, we have $\varphi(\sup x_n) = \sup \varphi(x_n)$.*

The next proposition gives another reason why the σ-weak topology is natural. Further, it gives a useful criterion to check σ-weak continuity in practice.

Proposition 1.1.3. *A functional $\varphi : M \to \mathbb{C}$ on a von Neumann algebra is σ-weakly continuous if and only if its restriction to the unit ball $(M)_1$ is weakly continuous.*

Remark 1.1.4. Similar results as in Proposition 1.1.2 hold for other types of maps, such as *-homomorphisms or more generally so called completely positive maps. In particular, conditional expectations, which will be introduced in Section 1.4.1, are of the latter kind.

1.2 Finite von Neumann algebras

A general von Neumann algebra is very difficult to understand. We are going to focus on *finite von Neumann algebras* whose definition will allows us to apply an abundance of Hilbert space arguments.

Definition 1.2.1 (Traces and finite von Neumann algebras)**.** Let M be a von Neumann algebra. A *normal tracial state* (or a *trace* for short) on M is a normal state $\tau : M \to \mathbb{C}$ such that $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

A von Neumann algebra M is *finite* if there exists a faithful family $(\tau_i)_i$ of traces on M, i.e. $\tau_i(x^*x) = 0$ for all i implies $x = 0$.

Remark 1.2.2. If M is a von Neumann algebra with a trace τ, then the GNS-construction gives rise to a representation $\pi : M \to \mathcal{B}(\mathrm{L}^2(M, \tau))$. There is a canonical *tracial vector* $\hat{1}$ in $\mathrm{L}^2(M)$. It satisfies $\langle \pi(xy)\hat{1}, \hat{1}\rangle = \tau(xy) = \tau(yx) = \langle \pi(yx)\hat{1}, \hat{1}\rangle$ for all $x, y \in M$. The representation π is σ-weakly continuous, since τ is so. One can show (using σ-weak compactness of the unit ball M_1) that $\pi(M) \subset \mathcal{B}(\mathrm{L}^2(M, \tau))$ is a von Neumann algebra and π is a closed map for σ-weak topologies on M and on $\mathcal{B}(\mathrm{L}^2(M, \tau))$ respectively. If we further assume that τ is faithful ($\tau(x^*x) = 0 \Rightarrow x = 0$), then π is injective and we can simply identify M its image $\pi(M)$. We will make use of this fact regularly.

Definition 1.2.3. A pair (M, τ) of a von Neumann algebra with a faithful tracial state is called a *tracial von Neumann algebra*. The norm $\|x\|_2 := \tau(x^*x)^{1/2}$, $x \in M$ is called the 2-norm on M.

If there is no confusion about τ is possible, we just call M a tracial von Neumann algebra.

Proposition 1.2.4. *The 2-norm of a tracial von Neumann algebra induces the strong topology on its unit ball. That is, if $(x_i)_i$ is a bounded net of elements of a tracial von Neumann algebras , then $x_i \to x$ strongly, if and only if $\|x - x_i\|_2 \to 0$.*

Proof. Let (M, τ) be a tracial von Neumann algebra. As explained in Remark 1.2.2, we may assume that M is represented on $\mathrm{L}^2(M)$. Since $\|x\|_2^2 = \tau(x^*x) = \langle x\hat{1}, x\hat{1}\rangle$, it is clear that strong convergence implies convergence in $\| \ \|_2$. Assume that $(x_i)_i$ is a bounded sequence converging to 0 in $\| \ \|_2$. First observe that $\|x_i^*\|_2^2 = \tau(x_i x_i^*) = \tau(x_i^* x_i) = \|x_i\|_2^2$. For $y \in M$, we have

$$\|x_i \hat{y}\|^2 = \tau(y^* x_i^* x_i y) = \tau(x_i y y^* x_i^*) = \|y^* \hat{x}_i^*\|^2 \leq \|y^*\|^2 \|x_i^*\|_2^2 = \|y\|^2 \|x_i\|_2^2 .$$

So $\|x_i \hat{y}\|_2^2 \to 0$. Now let $\varepsilon > 0$ and $\xi \in \mathrm{L}^2(M)$ an arbitrary unit vector. There is some $y \in M$ such that $\|\xi - \hat{y}\| < \varepsilon$. Further, $(x_i)_i$ is bounded, so there is $N > 0$ such that $\|x_i\| \leq N$ for all i. We obtain

$$\|x_i \xi\|^2 = \|x_i(\hat{y} + (\xi - \hat{y})\|^2 \leq \|x_i\|^2 \|\xi - \hat{y}\|^2 + \|x_i \hat{y}\|^2 \leq N^2 \varepsilon + \|x_i \hat{y}\|^2 \to N^2 \varepsilon .$$

Since ε was arbitrary and N is independent of ε, it follows that $\|x_i \xi\| \to 0$. This finishes the proof of the proposition. ☐

Remark 1.2.5. It is not true that the $\| \ \|_2$ and the strong topology agree on the whole of a von Neumann algebra. A counterexample can be found by considering the trace on $\mathrm{L}^\infty([0, 1])$, which is induced by the restricted Lebesgue measure.

Exercise 1.2.6. Find an unbounded sequence of elements in $\mathrm{L}^\infty([0, 1])$ which converge to 0 in $\| \ \|_2$, but not in the strong topology!

1.3 Group von Neumann algebras

Let Γ be a discrete group. Then the *left-regular representation* of Γ is $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ defined by $\lambda_g \delta_h = \delta_{gh}$. The group von Neumann algebra of Γ is

$$\mathrm{L}(\Gamma) = \lambda(\Gamma)'' = \overline{\mathbb{C}\Gamma}^{\mathrm{SOT}}. \qquad \text{(Proof the last equality!)}$$

Every group von Neumann algebra carries a natural trace $\tau(x) = \langle x\delta_e, \delta_e \rangle$. Indeed, by definition τ is weakly continuous (and hence strongly and σ-weakly continuous). So using continuity and linearity, it suffices to check that $\tau(u_g u_h) = \tau(u_h u_g)$ for all $g, h \in \Gamma$, which is easily done using the definition of τ.

In the following series of propositions we are going to show that τ is a faithful (Proposition 1.3.3).

Proposition 1.3.1. *Let $\varphi(x) = \langle x\xi, \xi \rangle$ be a vector state on a von Neumann algebra M. Then φ is faithful if and only if ξ is* separating *for M (i.e. $x\xi = 0 \Rightarrow x = 0$).*

Proof. The follows right from the equality $\varphi(x^*x) = \langle x^*x\xi, \xi \rangle = \langle x\xi, x\xi \rangle = \|x\xi\|$. □

In light of the previous proposition, we want to check that the vector δ_e is separating for the von Neumann algebra $\mathrm{L}(\Gamma)$. This is easily done using the following proposition.

Proposition 1.3.2. *Let $M \subset \mathcal{B}(H)$ be a von Neumann algebra. Then $\xi \in H$ is separating for M if and only if it is* cyclic *for M' (i.e. $M'\xi \subset H$ is dense).*

Proof. First assume that ξ is separating for M. Let $K := \overline{M'\xi}$. We have to show that $K = H$. Since K is invariant under M', the orthogonal projection $p_K : H \to K$ lies in $M = M''$. Since p_K is a projection, also $(1 - p_K)$ is a projection, implying that $(1 - p_K)^*(1 - p_K) = (1 - p_K)$. So $(1 - p_K)\xi = 0$ implies $(1 - p_K) = 0$. This shows that $p_K = 1$ and hence $K = H$.

Now assume that ξ is cyclic for M'. Take $x \in M$ such that $x\xi = 0$. We have to show that $x = 0$. Since $x \in M$, we have $0 = M'x\xi = xM'\xi$. Since ξ is cyclic for M', it follows that $xH = 0$. But this shows that $x = 0$, finishing the proof. □

Now finally let us check that δ_e is indeed cyclic for the commutant of $\mathrm{L}(\Gamma)$. To this end, consider the right regular representation $\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ defined by $\rho(g)\delta_h = \delta_{hg}$. Note that this is a right representation (i.e. $\rho(gh) = \rho(h)\rho(g)$). It is easy to see that $\mathrm{R}(\Gamma) := \rho(\Gamma)'' \subset \mathrm{L}(\Gamma)'$. Since $\mathrm{R}(\Gamma)\delta_e \supset \mathbb{C}\Gamma$ is dense in $\ell^2(\Gamma)$, it follows that δ_e is a separating vector for $\mathrm{L}(\Gamma)$. The previous discussion proves the following statement.

Proposition 1.3.3. *Let Γ be a discrete group. The vector state $\tau = \langle \cdot\, \delta_e, \delta_e \rangle$ defines a faithful normal tracial state on the group von Neumann algebra $\mathrm{L}(\Gamma)$.*

Remark 1.3.4. Let Γ be a discrete group and τ the natural trace on its group von Neumann algebra $\mathrm{L}(\Gamma)$. Then the GNS-representation of $\mathrm{L}(\Gamma)$ with respect to τ is given by $\mathrm{L}(\Gamma) \subset \mathcal{B}(\ell^2\Gamma)$ with cyclic vector δ_e. Indeed, the fact that τ is the vector state on $\mathrm{L}(\Gamma)$ associated with the cyclic vector δ_e suffices to apply the uniqueness part of the GNS-theorem.

One important tool when studying group von Neumann algebras is the fact that we can consider Fourier expansions of arbitrary elements in a group von Neumann algebra.

Definition 1.3.5 (Fourier coefficients)**.** Let Γ be a discrete group and $x \in \mathrm{L}(\Gamma)$. Then $x_g := \tau(x u_g^*)$ is called the *g-th Fourier coefficient* of x.

Proposition 1.3.6. *Let Γ be a discrete group and $x \in \mathrm{L}(\Gamma)$ then $x = \sum_{g \in G} x_g u_g$ where the sum converges in $\| \ \|_2$.*

Proof. For $x \in \mathrm{L}(\Gamma)$ note that $x\delta_e = \sum_{g \in G} x_g \delta_g$. For $F \subset \Gamma$ finite write $x_F = \sum_{g \in F} x_g u_g$. We have

$$(x - x_F)\delta_e = \sum_{g \in \Gamma \setminus F} x_g \delta_g .$$

Hence

$$\|x - x_F\|_2 = \|(x - x_F)\delta_e\| = \| \sum_{g \in \Gamma \setminus F} x_g \delta_g \| .$$

This converges to 0, since $x\delta_e = \sum_{g \in \Gamma} x_g \delta_g$ is 2-summable. This leads to the desired conclusion. □

1.4 The group-measure space construction

1.4.1 Conditional expectations

In this section we are going to introduce an important tool in von Neumann algebras. It is the basic mean to link the members of an inclusion $N \subset M$ of von Neumann algebras. In the definition of the group-measure space construction it will play a crucial role.

Definition 1.4.1 (Conditional expectation)**.** Let $N \subset M$ be an inclusion of von Neumann algebras. A *conditional expectation* from M onto N is a unital norm contraction $\mathrm{E} : M \to N$ satisfying $\mathrm{E}(n_1 m n_2) = n_1 \mathrm{E}(m) n_2$ for all $n_1, n_2 \in N$ and $m \in M$. A conditional expectation E is called normal if $\mathrm{E}(\sup x_i) = \sup \mathrm{E}(x_n)$ for all bounded monotone sequences $x_1 \leq x_2 \leq \cdots \leq \lambda 1$ of self-adjoint elements in M.

Remark 1.4.2. A conditional expectation is normal if and only if it is σ-weakly continuous.

Example 1.4.3. Let $\Lambda \leq \Gamma$ be an inclusion of groups. Then $\mathrm{L}(\Lambda) \subset \mathrm{L}(\Gamma)$ and there is a normal conditional expectation $\mathrm{E} : \mathrm{L}(\Gamma) \to \mathrm{L}(\Lambda)$ satisfying $\mathrm{E}(u_g) = u_g \mathbb{1}_\Lambda(g)$ for all $g \in \Gamma$.

Exercise 1.4.4. Let $\Lambda \trianglelefteq \Gamma$ be a normal inclusion of groups. Show that for every trace τ on $\mathrm{L}(\Lambda)$ the composition $\tau \circ \mathrm{E}$ is a trace on $\mathrm{L}(\Gamma)$.

An important property of conditional expectations is the fact that they preserve positivity. This is the content of the next proposition.

Proposition 1.4.5. *Let $\mathrm{E} : M \to N$ be a conditional expectation between von Neumann algebras. Then E is positive, meaning that $\mathrm{E}(x) \geq 0$ for all $x \geq 0$ in M.*

Proof. Let $x \geq 0$ be a positive element in M. If φ is a state on N, then $\varphi \circ \mathrm{E}$ is a unital contractive state on M, so it is positive. Hence $\varphi(\mathrm{E}(x)) = \varphi \circ \mathrm{E}(x) \geq 0$. This shows that $\mathrm{E}(x) \geq 0$ in N. □

Definition 1.4.6. Let $\mathrm{E} : M \to N$ be a conditional expectation between von Neumann algebras. Then E is called faithful if $\mathrm{E}(X^* x) = 0$ implies $x = 0$.

Proposition 1.4.7. *Let $\mathrm{E} : M \to N$ be a conditional expectation between von Neumann algebras and let φ be a faithful normal state on N. Then $\varphi \circ \mathrm{E}$ is a faithful normal state on M.*

Proof. From Proposition 1.4.5 we know that $\varphi \circ \mathrm{E}$ is a state for every state φ on N. So we have to show faithfulness, assuming φ and E are faithful. Let $x \in M$ and assume that $\varphi \circ \mathrm{E}(x^*x) = 0$. Since $\mathrm{E}(x^*x) \geq 0$ we find $y \in N$ such that $\mathrm{E}(x^*x) = y^*x$. Faithfulness of φ implies that $y = 0$ and hence $\mathrm{E}(x^*x) = 0$. We can now conclude by applying faithfulness of E. ☐

1.4.2 Actions on standard probability measure spaces

Definition 1.4.8. A probability space (X, μ) is called *standard probability space* if the σ-algebra of X, up to negligible sets, arises as the Borel σ-algebra of a Polish space (i.e. a separable completely metrisable space).

If (X, μ) is a standard probability space, then $\mathrm{Aut}(X, \mu)$ is the group of measure class preserving Borel bijections, identifying two automorphisms if they agree on a set of full measure.

If Γ is a discrete group, then an action $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ is a homomorphism $\Gamma \to \mathrm{Aut}(X, \mu)$. We call this action

- probability measure preserving, if there is a probability measure ν on X which is equivalent to μ (i.e. a set $N \subset X$ is μ-negligible if and only if it is ν-negligible) such that every automorphism in α_γ preserves ν;
- essentially free if for every $\gamma \in \Gamma$ the set $X_\gamma = \{x \in X \mid \alpha_\gamma(x) = x\}$ is negligible;
- ergodic if every Γ-invariant Borel subset of X is either negligible or co-negligible.

Remark 1.4.9. Every standard probability measure space is isomorphic to $[0,1] \cup D$ with the restricted Lebesgue measure and an atomic measure on the countable set D. We speak about standard probability measure spaces though, in order to naturally include examples such as the product space $\{0,1\}^\Gamma$ in our thinking.

Remark 1.4.10. We usually denote standard probability spaces by X, suppressing the notation of the measure μ. Further, 'probability measure preserving' is abbreviated 'pmp'. For simplicity, we refer to 'essentially free' actions as just 'free'. In this lecture free ergodic pmp actions of discrete groups $\Gamma \curvearrowright X$ will play a principal role.

Example 1.4.11. Let Γ be a discrete group and X_0 some standard probability space. Then Γ acts on the product measure space $X = X_0^\Gamma$ by shifting indices $(gf)(h) = f(g^{-1}h)$. This action is called the *Bernoulli shift of* Γ *with base space* X_0. It is always probability measure preserving. If Γ is an infinite group and (X_0, μ_0) is non-trivial (i.e. it is not isomorphic to one point), then every Γ-Bernoulli shift is free and ergodic.

Exercise 1.4.12. Let Γ be an infinite discrete group and (X_0, μ_0) a non-trivial standard probability measure space. Prove that $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ is free and ergodic.

Example 1.4.13. Let $\Gamma \leq G$ be a dense subgroup of a compact second countable group. Denote by μ the normalised Haar measure on G. Then $\Gamma \curvearrowright (G, \mu)$ is a free ergodic pmp action.

Exercise 1.4.14. Prove that in the situation of Example 1.4.13, the action $\Gamma \curvearrowright (G, \mu)$ is free ergodic and pmp.

1.4.3 Fell's absorption property

Theorem 1.4.15. *Let Γ be a discrete group and $U : \Gamma \to \mathcal{U}(H)$ a unitary representation of Γ. Then $U \otimes \lambda \cong \bigoplus_{\dim H} \lambda$.*

Proof. We show that the map $W : H \otimes \ell^2(\Gamma) \to H \otimes \ell^2(\Gamma)$ satisfying $W(\xi \otimes \delta_g) = U_g\xi \otimes \delta_g$ is a well-defined unitary. Indeed, for finite linear combinations $\sum_{g\in\Gamma} \xi_g \otimes \delta_g$ we obtain

$$\|W(\sum_{g\in\Gamma} \xi_g \otimes \delta_g)\|^2 = \|\sum_{g\in\Gamma} U_g\xi_g \otimes \delta_g\|^2 = \sum_{g\in\Gamma} \|U_g\xi_g\|^2 = \sum_{g\in\Gamma} \|\xi_g\|^2 = \|\sum_{g\in\Gamma} \xi_g \otimes \delta_g\|^2 .$$

So W extends to an isometry on $H \otimes \ell^2(\Gamma)$ whose image is dense. This means that W is a unitary.

If $\xi \in H$ and $h \in \Gamma$, we check that

$$\begin{aligned}(U_g \otimes \lambda_g)W(\xi \otimes \delta_h) &= (U_g \otimes \lambda_g)(U_h\xi \otimes \delta_h) \\ &= U_{gh}\xi \otimes \delta_{gh} \\ &= W(\xi \otimes \delta_{gh}) \\ &= W(\mathrm{id} \otimes \lambda_g)(\xi \otimes \delta_h) .\end{aligned}$$

$W(\mathrm{id} \otimes \lambda) = (U \otimes \lambda)W$, finishing the proof of the theorem. □

Corollary 1.4.16. *Let Γ be a discrete group and $U : \Gamma \to \mathcal{U}(H)$ some unitary representation of Γ. Then the representation $U \otimes \lambda : \Gamma \to \mathcal{U}(H \otimes \ell^2\Gamma)$ extends to a strongly continuous representation of $\mathrm{L}(\Gamma)$ on $H \otimes \ell^2(\Gamma)$.*

Proof. By Fell's absorption property 1.4.15 we may assume that U is the trivial representation on H. We obtain a *-representation of the group ring $\pi : \mathbb{C}\Gamma \to \mathcal{B}(H \otimes \ell^2\Gamma)$ extending $\mathrm{id} \otimes \lambda$ linearly. If for every net $(x_i)_i$ in $\mathbb{C}\Gamma$ converging to 0 in the σ-weak topology of $\mathrm{L}(\Gamma)$ also the image $(\pi(x_i))_i$ converges to 0 in the σ-weak topology of $\mathcal{B}(H \otimes \ell^2\Gamma)$, then π extends to $\mathrm{L}(\Gamma)$.

So let $(x_i)_i$ be net in $\mathbb{C}\Gamma$ converging to 0 σ-weakly. Let $(\xi_j)_j$ and $(\eta_j)_j$ be 2-summable sequences in $H \otimes \ell^2\Gamma$. Chose some orthonormal basis $(e_n)_n$ of H and write $\xi_j = \sum_n e_n \otimes \xi_{j,n}$, $\eta_j = \sum_n e_n \otimes \eta_{j,n}$. Then the sequences $(\xi_{j,n})_{j,n}$ and $(\eta_{j,n})_{j,n}$ are 2-summable. Further

$$\begin{aligned}\sum_j \langle \pi(x_i)\xi_j, \eta_j\rangle &= \sum_j \langle \sum_n e_n \otimes x_i\xi_{j,n}, \sum_n e_n \otimes \eta_{j,n}\rangle \\ &= \sum_{j,n} \langle x_i\xi_{j,n}, \eta_{j,n}\rangle \\ &\to 0 .\end{aligned}$$

This finishes the proof of the corollary. □

1.4.4 Construction of the group-measure space construction

Definition 1.4.17 (Group-measure space construction)**.** Let $\Gamma \curvearrowright X$ be a measure class preserving action on a standard probability measure space. The group-measure space construction of $\Gamma \curvearrowright X$ is the unique von Neumann algebra $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ such that

- $\mathrm{L}^\infty(X) \subset M$,

- $\mathrm{L}(G) \subset M$,
- M is generated by $\mathrm{L}^\infty(X)$ and $\mathrm{L}(G)$,
- $u_g x u_g^* = {}^g x$, and
- there is a faithful normal conditional expectation $\mathrm{E} : M \to \mathrm{L}^\infty(X)$ such that $\mathrm{E}(u_g) = \delta_{g,e}$.

Proposition 1.4.18. *Let $\Gamma \curvearrowright (X, \mu)$ be probability measure preserving action on a standard probability measure space. Then the group-measure space construction $\mathrm{L}^\infty(X) \rtimes \Gamma$ exists.*

Proof. Replacing μ by some equivalent probability measure, we may assume that $\Gamma \curvearrowright X$ preserves μ. Then the action of Γ induces a unitary representation on $\mathrm{L}^2(X, \mu)$ by $f \mapsto {}^g f$. (Here ${}^g f$ denotes the function sending h to $f(g^{-1}h)$.) Denote this representation by $g \mapsto U_g$. Then we obtain a representation of Γ on $\mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma)$ by $g \mapsto U_g \otimes \lambda_g$. Corollary 1.4.16 says that $U \otimes \lambda$ extends to a representation of $\mathrm{L}(\Gamma)$ on $\mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma)$. Now represent $\mathrm{L}^\infty(X)$ on $\mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma)$ as $f \mapsto m_f \otimes 1$. It suffices to check that the action of $u_g f$ and ${}^g f u_g$ on elementary tensors $\xi \otimes \delta_h \in \mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma)$ agree, so as to conclude that $u_g f u_g^* = {}^g f$ in the von Neumann algebra $M := (\mathrm{L}(\Gamma) \cup \mathrm{L}^\infty(X))'' \subset \mathcal{B}(\mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma))$. This is verified by a short calculation.

It remains to construct a faithful normal conditional expectation $M \to \mathrm{L}^\infty(X)$. To this end consider the coisometry $V : \mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma) \to \mathrm{L}^2(X, \mu)$ projecting on $\mathrm{L}^2(X, \mu) \otimes \delta_e$ and identify the its image with $\mathrm{L}^2(X, \mu)$. Then $V f u_g V^* = \delta_{g,e} f$ for all $f \in \mathrm{L}^\infty(X)$ and $g \in \Gamma$. Note that we identify the two representations of $\mathrm{L}^\infty(X)$ on $\mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma)$ and $\mathrm{L}^2(X, \mu)$, respectively. Let $\iota : \mathrm{L}^\infty(X) \to M$ be the inclusion map. Then $\mathrm{E} := \iota \circ \mathrm{Ad}\, V$ is a well-defined map from M onto $\mathrm{L}^\infty(X)$. By construction it is contractive and it satisfies

$$\mathrm{E}(f_1 f u_g f_2) = \iota \circ V(f_1 f u_g f_2) = \iota(V f_1 V^* V f u_g V^* V f_2 V) = \mathrm{E}(f_1)\mathrm{E}(f u_g)\mathrm{E}(f_2) = f_1 \mathrm{E}(f u_g) f_2 \,,$$

for all $f_1, f_2, f \in \mathrm{L}^\infty(X)$ and all $g \in \Gamma$. Linearity hence shows that E is a conditional expectation. Note that E is normal as a composition of two normal maps. So it remains to show faithfulness of E. We follow the strategy of the proof of Proposition 1.3.3. If $x \in M$ satisfies $\mathrm{E}(x^*x) = 0$, then $V x^* x V^* = 0$. Since $V^* \xi = \xi \delta_e$ for every $\xi \in \mathrm{L}^2(X, \mu)$, this implies $x(\xi \otimes \delta_e) = 0$ for every $\xi \in \mathrm{L}^2(X, \mu)$. Now consider the right regular representation $\mathrm{id} \otimes \rho : \Gamma \to \mathcal{U}(\mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma))$. Since $(\mathrm{id} \otimes \rho)(g)$ commutes with M for every $g \in \Gamma$, we see that $x(\xi \otimes \delta_g) = 0$ for every $g \in \Gamma$. Since vectors of this form span a dense subspace of $\mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma)$, it follows that $x = 0$. So E is a faithful conditional expectation. □

Remark 1.4.19. The group measure space construction of an arbitrary measure class preserving action exists. The poof of this is slightly more complicated than the one of Proposition 1.4.18.

We are next going to prove that the group measure-space construction is uniquely up to isomorphism. To this end, we employ the GNS-construction with respect to a natural trace on $\mathrm{L}^\infty(X) \rtimes \Gamma$.

Proposition 1.4.20. *Let $\Gamma \curvearrowright (X, \mu)$ preserve the probability measure μ. Let τ_0 be the trace on $\mathrm{L}^\infty(X)$ defined by integrating against μ. Let M be a group-measure space construction for $\Gamma \curvearrowright X$. Then $\tau_0 \circ \mathrm{E} : M \to \mathbb{C}$ is a faithful trace on M.*

Proof. Proposition 1.4.7 implies that $\tau = \tau_0 \circ \mathrm{E}$ is a faithful normal state on M. We only need to prove that τ is tracial. To this end we have to check that $\tau(mn) = \tau(nm)$ for all $m, n \in M$. By linearity and continuity, we may assume that $m = m_g u_g$ and $n = n_h u_h$ for $m_g, n_h \in \mathrm{L}^\infty(X)$ and $g, h \in \Gamma$.

$$\mathrm{E}(mn) = \mathrm{E}(m_g u_g n_h u_h) = \mathrm{E}(m_g \, {}^g n_h u_{gh}) = \delta_{g,h^{-1}} m_g \, {}^g n_h$$

and

$$\mathrm{E}(nm) = \mathrm{E}(n_h u_h m_g u_g) = \mathrm{E}(n_h\, {}^h m_g u_{hg}) = \delta_{g,h^{-1}} n_h\, {}^h m_g\,.$$

Since τ_0 is defined by integration against the Γ-invariant measure μ, we obtain

$$\begin{aligned}
\tau(mn) &= \delta_{g,h^{-1}} \int_X (m_g\, {}^g n_h)(x) \mathrm{d}\mu(x) \\
&= \delta_{g,h^{-1}} \int_X m_g(x) n_h(g^{-1}x) \mathrm{d}\mu(x) \\
&= \delta_{g,h^{-1}} \int_x m_g(x) n_h(hx) \mathrm{d}\mu(x) \\
&= \delta_{g,h^{-1}} \int_x m_g(h^{-1}x) n_h(x) \mathrm{d}\mu(x) \\
&= \delta_{g,h^{-1}} \int_X ({}^h m_g n_h)(x) \mathrm{d}\mu(x) \\
&= \tau(nm)\,.
\end{aligned}$$

This is what we had to show. □

Theorem 1.4.21. *Let $\Gamma \curvearrowright (X,\mu)$ preserve the probability measure μ. Let τ be the natural trace on the group-measure space construction M of $\Gamma \curvearrowright X$. The GNS-construction of M with respect to τ is given as follows:*

- *The GNS-Hilbert space is $H_\tau \cong \mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$,*
- *$\mathrm{L}^\infty(X)$ acts on $\mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$ by multiplication operators in the first tensor factor,*
- *$\Gamma \subset \mathrm{L}(\Gamma)$ acts via $U \otimes \lambda$ on $\mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$, where U is the unitary representation of Γ induced by the measure preserving action $\Gamma \curvearrowright X$, and*
- *the cyclic vector is $\mathbb{1}_X \otimes \delta_e \in \mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$.*

In particular, the group-measure space construction is unique up to isomorphism preserving the inclusions $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ and $\mathrm{L}(\Gamma) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$.

Proof. Let (M,τ) be given as in the statement of the theorem. Let $\mathcal{M} \subset M$ be the *-subalgebra generated by $\mathrm{L}^\infty(X)$ and Γ. Note that every element in $\mathcal{M}$ is a linear combination of xu_g, with $x \in \mathrm{L}^\infty(X)$ and $g \in \Gamma$.

- In order to simplify notation, we consider $\mathcal{M}$ as a subspace of the GNS-Hilbert space H_τ. If $g \neq h$ are different elements from Γ, then $xu_g \perp yu_h$ with respect to τ for all $x, y \in \mathrm{L}^\infty(X)$. Indeed,

  $$\langle xu_g, yu_h\rangle = \tau((yu_h)^* xu_g) = \tau(y^* x u_g u_h^*) = \tau \circ \mathrm{E}(y^* x u_g u_h^*) = \delta_{g,h}\tau(y^* x) = \delta_{g,h}\langle x, y\rangle_{\mathrm{L}^2(X,\mu)}$$

  So the map $W : \mathcal{M} \to \mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$ satisfying $W(xu_g) = \hat{x} \otimes \delta_g$ extends to a well-defined isometry $\mathrm{L}^2(M,\tau) \to \mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$. Since W has dense range, it is a unitary. We showed that $H_\tau \cong \mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$ via the unitary W.

- Now let $x, y \in \mathrm{L}^\infty(X)$ and $g \in \Gamma$. Then

$$Wx(\widehat{yu_g}) = W\widehat{xyu_g} = \widehat{xy} \otimes \delta_g = (x \otimes \mathrm{id})(\hat{y} \otimes \delta_g) = (x \otimes \mathrm{id})W\widehat{yu_g}\,.$$

  So $WxW^* = x \otimes \mathrm{id}$ on $\mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$.

- Now let $x \in \mathrm{L}^\infty(X)$ and $g, h \in \Gamma$. We obtain

$$Wu_g\widehat{xu_h} = W\,\widehat{{}^g x u_{gh}} = \,\widehat{{}^g x} \otimes \delta_{gh} = (U_g \otimes \lambda_g)W\widehat{xu_h}\,.$$

  This shows that $Wu_gW^* = U_g \otimes \lambda_g$ on $\mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$.

- Since $W\hat{1} = \mathbb{1}_X \otimes \delta_e$, this cyclic vector in $\mathrm{L}^2(X,\mu) \otimes \ell^2(\Gamma)$.

The uniqueness of the group-measure space construction is now clear from the first part of the theorem. □

The group-measure space construction is uniquely determined up to an isomorphism preserving the inclusions $\mathrm{L}^\infty(X), \mathrm{L}(\Gamma) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$. Indeed, if M is a group-measure space construction, associated with a pmp action $\Gamma \curvearrowright X$, then we can find a trace τ on M satisfying $\tau = \tau \circ \mathrm{E}$. The GNS-construction with respect to such a state gives the representation that we constructed in the previous proposition.

Definition 1.4.22 (Fourier coefficients)**.** Let $\Gamma \curvearrowright X$ be a pmp action of a discrete group and let $x \in \mathrm{L}^\infty(X) \rtimes \Gamma$. Then $x_g := \mathrm{E}(xu_g^*)$ is called the g-th Fourier coefficient of x.

The next proposition is proved in a similar fashion to its analogue for group von Neumann algebras. (Proposition 1.3.6)

Proposition 1.4.23. *Let $\Gamma \curvearrowright X$ be a pmp action of a discrete group and let $x \in \mathrm{L}^\infty(X) \rtimes \Gamma$. The sum $\sum_{g\in\Gamma} x_g u_g$ converges in $\|\ \|_2$ to x in $\mathrm{L}^\infty(X) \rtimes \Gamma$.*

1.4.5 Cartan subalgebras

Definition 1.4.24 (MASAs)**.** A maximal abelian von Neumann subalgebra $A \subset M$ is called *MASA*.

Remark 1.4.25. An abelian von Neumann subalgebra $A \subset M$ satisfies $A' \cap M \supset A$. Now A is a MASA in M if and only if $A' \cap M = A$.

Definition 1.4.26 (Normaliser)**.** Let $N \subset M$ be an inclusion of von Neumann algebras. Then $\mathcal{N}_M(N) = \{u \in \mathcal{U}(M) \mid uNu^* = N\}$ is called the *group of normalising unitaries* of N in M. The von Neumann algebra $\mathcal{N}_M(N)''$ generated by this group is called the *normaliser* of N in M. We call $N \subset M$ a regular inclusion if $\mathcal{N}_M(N)'' = M$.

Definition 1.4.27 (Cartan subalgebra)**.** Let M be a finite von Neumann algebra. A MASA $A \subset M$ is called *Cartan subalgebra* of M if its normaliser equals M.

Proposition 1.4.28. *Let $\Gamma \curvearrowright X$ be a free pmp action on a standard probability space. Then $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ is a Cartan subalgebra.*

Proof. Let $\Gamma \curvearrowright X$ be a free pmp action. Write $A := \mathrm{L}^\infty(X)$ and $M := \mathrm{L}^\infty(X) \rtimes \Gamma$.

Note that A is generated by its unitaries. So $(\mathcal{U}(A) \cup \{u_g \mid g \in \Gamma\})''$ contains A and $\mathrm{L}(G)$ and it hence equals M. So $\mathcal{N}_M(A)'' \supset (\mathcal{U}(A) \cup \{u_g \mid g \in \Gamma\})'' = M$.

It remains to check that $A \subset M$ is a MASA. To this end let $x \in A' \cap M$. We can write $x = \sum_{g \in \Gamma} x_g u_g$. Assume that $x_g \neq 0$ for some $g \neq e$. Let $U \subset \operatorname{supp} x_g$ be a non-negligible subset such that $gU \cap U = \varnothing$ and let $p := \mathbb{1}_U \in A$ be the associated non-zero projection. Then

$$\sum_g \mathbb{1}_U x_g u_g = px = xp = \sum_g x_g u_g \mathbb{1}_U = \sum_g x_g \mathbb{1}_{gU} u_g \,.$$

So the g-th Fourier coefficient of px equals $x_g \mathbb{1}_U = x_g \mathbb{1}_{gU}$. Since $gU \cap U = \varnothing$, this is a contradiction. We showed that $A' \cap M = A$, finishing the proof of the proposition. □

Remark 1.4.29. A Cartan subalgebra of a finite von Neumann algebra arising from a group-measure space construction as in the previous proposition is called *group-measure space Cartan subalgebra*.

1.4.6 Orbit equivalence and Cartan preserving isomorphisms

Definition 1.4.30 (Orbit equivalence)**.** Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be two free actions on standard probability measure spaces. They are called *orbit equivalent* if there is an isomorphism $\Delta : X \cong Y$ of such that $\Lambda\Delta(x) = \Delta(\Gamma x)$ for almost every $x \in X$.

Definition 1.4.31 (Cocycle)**.** Let $\Gamma \curvearrowright X$ be an action on a standard probability measure space and let Λ be a group. A measurable map $c : \Gamma \times X \to \Lambda$ is called a cocycle, if

$$c(gg', x) = c(g, g'x)c(g', x)$$

for all $g, g' \in \Gamma$ and almost every $x \in X$.

Proposition 1.4.32. *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be free actions of countable groups on standard probability measure spaces and let $\Delta : X \to Y$ be an orbit equivalence between these two actions. Then*

$$c(g, x)\Delta(x) = \Delta(gx) \qquad g \in \Gamma$$

defines an almost everywhere well-defined cocylce $\Gamma \times X \to \Lambda$.

Proof. We first show that the equation $c(g, x)\Delta(x) = \Delta(gx)$ gives rise to a well-defined measurable map $c : \Gamma \times X \to \Lambda$. Fix $g \in \Gamma$. Then

$$\{x \in X \mid \exists h \neq h' : \Delta(gx) = h\Delta(x) = h'\Delta(x)\} \subset \bigcup_{h \neq e} \Delta^{-1}\{y \in Y \mid hx = x\}$$

is a countable union of negligible sets and it is hence negligible itself. So, up to negligible sets, c is well-defined. Denoting by α the action of Γ and by β the action of Λ, we prove measurability of c. For $g \in \Gamma$ and $h \in \Lambda$ fixed, $\{x \in X \mid c(g, x) = h\}$ is the set where α_g and $\Delta^{-1} \circ \beta_h \circ \Delta$ agree. Since both these maps are measurable, it follows that $\{x \in X \mid c(g, x) = h\}$ is measurable. Since Γ and Λ are discrete, we conclude that c is measurable.

Let deal with now deal with measure theoretic problems, making use of the fact that Γ and Λ are countable. Considering a conegligible subset of X, we may assume that $c(g, x)\Delta(x) = \Delta(gx)$ for all

$x \in X$ and that $\Lambda \curvearrowright X$ is free in the set-theoretic sense of the word. Then we obtain for $x \in X$ and $g, g' \in \Gamma$ that

$$c(g, g'x)c(g', x)\Delta(x) = c(g, g'x)\Delta(g'x) = \Delta(gg'x) = c(gg', x)\Delta(x) .$$

By freeness of $\Lambda \curvearrowright Y$, this shows that $c(g, g'x)c(g', x) = c(gg', x)$. This finishes the proof of the proposition. □

Definition 1.4.33 (Orbit equivalence cocycle)**.** The cocycle defined in the last proposition is called an *orbit equivalence cocycle* for $(\Gamma \curvearrowright X) \sim_{\mathrm{OE}} (\Lambda \curvearrowright Y)$.

Theorem 1.4.34. *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be orbit equivalent free pmp actions of countable groups. Then there is an isomorphism $\varphi : \mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(Y) \rtimes \Lambda$ such that $\varphi(\mathrm{L}^\infty(X)) = \mathrm{L}^\infty(Y)$.*

Proof. Let $\Delta : X \to Y$ be an orbit equivalence between $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$. Introducing the action $\lambda x := \Delta^{-1}(\lambda\Delta(x))$ on X, we may assume that $X = Y$ and $\Delta = \mathrm{id}$. Let $c : \Gamma \rtimes X \to \Lambda$ be the orbit equivalence cocycle.

For $g \in \Gamma$, $h \in \Lambda$ let $A_g^h := \{x \in X \mid c(g, x) = h\}$ and consider the sum $v_g = \sum_h u_h \mathbb{1}_{A_g^h}$. Since the sets $(A_g^h)_h$ and $(hA_g^h)_h = (gA_g^h)_h$ are pairwise disjoint, $\sum_h u_h \mathbb{1}_{A_g^h}$ is a sum of partial isometries with pairwise orthogonal support and image projection. Hence $\sum_h u_h \mathbb{1}_{A_g^h}$ converges in the SOT and $v_g \in \mathrm{L}^\infty(X) \rtimes \Lambda$ is well-defined.

We show that v_g is a unitary for every g. This follows from the calculation

$$v_g v_g^* = \sum_h u_h \mathbb{1}_{A_g^h} u_h^* = \sum_h \mathbb{1}_{hA_g^h} = \sum_h \mathbb{1}_{gA_g^h} = 1 ,$$

and the fact that $\tau(v_g^* v_g) = \tau(v_g v_g^*) = 1$. Here τ denotes the natural trace of $\mathrm{L}^\infty(X) \rtimes \Lambda$.

The map $g \mapsto v_g$ defines a unitary representation of Γ inside $\mathrm{L}^\infty(X) \rtimes \Lambda$: we have

$$v_g v_{g'} = \sum u_h \mathbb{1}_{A_g^h} u_{h'} \mathbb{1}_{A_{h'}^{g'}} = \sum u_{hh'} \mathbb{1}_{(h')^{-1}A_g^h} \mathbb{1}_{A_{h'}^{g'}} = \sum u_{hh'} \mathbb{1}_{(h')^{-1}A_g^h \cap A_{h'}^{g'}}$$

and

$$(h')^{-1}A_g^h \cap A_{h'}^{g'} = \{x \in X \mid c(g, h'x) = h \text{ and } c(g', x) = h'\} \subset \{x \in X \mid \underbrace{c(g, h'x)c(g', x)}_{=c(gg', x)} = hh'\} .$$

This implies $v_g v_{g'} = v_{gg'}$.

Let $\mathrm{E} : \mathrm{L}^\infty(X) \rtimes \Lambda \to \mathrm{L}^\infty(X)$ be the natural conditional expectation. We obtain

$$\mathrm{E}(v_g) = \sum_h \mathrm{E}(u_h) \mathbb{1}_{A_h^g} = \sum_h \tau(u_h) \mathbb{1}_{A_h^g} = \mathbb{1}_{A_e^g} .$$

If $c(g, x) = e$, then $gx = c(g, x)x = x$. Because of freeness of $\Lambda \curvearrowright X$, this implies A_e^g is a negligible set if $g \neq e$ and A_e^e is a conegligible set. So $\mathrm{E}(v_g) = \delta_{g,e}1$. In particular, we have $\tau(v_g) = \delta_{g,e}$, so that $g \mapsto v_g$ extends to a representation of $\mathrm{L}(\Gamma)$ inside $\mathrm{L}^\infty(X) \rtimes \Lambda$. Summarising we have found $\mathrm{L}^\infty(X), \mathrm{L}(\Gamma) \subset \mathrm{L}^\infty(X) \rtimes \Lambda$ and the natural conditional expectation $\mathrm{E} : \mathrm{L}^\infty(X) \rtimes \Lambda$ restricts to the natural trace of $\mathrm{L}(\Gamma)$. In order to conclude that the map $\varphi : \mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(X) \rtimes \Lambda$ satisfying $\varphi|_{\mathrm{L}^\infty(X)} = \mathrm{id}$ and $\varphi(u_g) = v_g$ is a well-defined isomorphism preserving the group-measure space Cartan subalgebra, it remains to show that $\mathrm{Ad}\, v_g|_{\mathrm{L}^\infty(X)}$ implements the action $\Gamma \curvearrowright X$. So let $a \in \mathrm{L}^\infty(X)$ and $g \in \Gamma$. Then

$$v_g a v_g^* = \sum_{h,h'} u_h \mathbb{1}_{A_h^g} a \mathbb{1}_{A_{h'}^{g'}} u_{h'}^* = \sum_h v_h \mathbb{1}_{A_h^g} a v_h^* = \sum_h \mathbb{1}_{hA_h^g} ha = \sum_h \mathbb{1}_{gA_h^g} ga = ga .$$

This finishes the proof of the theorem. □

We are going to prove the converse to the last theorem. Let us explain the strategy of proof and assume that $\mathrm{L}^\infty(X) \rtimes \Gamma \cong \mathrm{L}^\infty(X) \rtimes \Lambda$ in a Cartan preserving way for two free pmp actions of Γ and Λ on X. The most essential information that we obtain through this isomorphism is that there are unitaries v_h, $h \in \Lambda$ which normalise the Cartan subalgebra $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$. Since this is the most important information at hand, we need to study in full generality the normaliser $\mathcal{N}_{\mathrm{L}^\infty(X)\rtimes\Gamma}(\mathrm{L}^\infty(X))$, which is done in Proposition 1.4.36. To this end, we need to do some technical work first.

Proposition 1.4.35. *Let A be a separable abelian von Neumann algebra. Then there is a standard probability space X such that $A \cong \mathrm{L}^\infty(X)$. Further, if $\varphi \in \mathrm{Aut}(\mathrm{L}^\infty(X))$, there is a measurable isomorphism $\alpha : X \to X$ such that $\varphi = \alpha^*$, i.e. $\varphi(a) = a \circ \alpha$ for all $a \in \mathrm{L}^\infty(X)$.*

Proof. Let A be a separable abelian von Neumann algebra and $\varphi \in \mathrm{Aut}(A)$. By definition of separability, there is a Hilbert space of countable dimension H such that $A \subset \mathcal{B}(H)$. Taking a countable orthonormal basis of H, we obtain a countable faithful family of tracial vector states $(\tau_n)_n$ on A. Then $\tau := \sum_{n\in\mathbb{N}^\times} 2^{-n}\tau_n$ is a faithful trace on A. From now on, $\| \ \|_2$ denotes the 2-norm induced by τ on A. By construction of τ, we obtain an embedding $\mathrm{L}^2(A,\tau) \subset H^{\oplus\infty}$ into a Hilbert space of countable dimension. We can hence find a countable $\| \ \|_2$-dense D set of A. Further, we may assume that $1 \in D$ and that D is φ-invariant. Let $B = \mathrm{C}^*(D)$ be the abelian C^*-algebra generated by D. Then B is φ-invariant. We hence find a compact second countable space X such that $B \cong \mathrm{C}(X)$. The automorphism $\varphi|_B$ defines a homeomorphism $\alpha : X \to X$. Further, $\tau|_B$ defines a faithful state, and hence a Borel probability measure μ on X with full support. The GNS-representation of τ gives an embedding $B \hookrightarrow \mathcal{B}(\mathrm{L}^2(X,\mu))$ such that $\mathrm{L}^\infty(X,\mu) = \overline{B}^{\mathrm{SOT}} = A$. Since $\alpha^* = \varphi|_B$ extends to a well-defined automorphism of $\mathrm{L}^\infty(X,\mu)$, it follows that α preserves the measure class of μ and $\alpha^* = \varphi$. ☐

Proposition 1.4.36. *Let $\Gamma \curvearrowright X$ be a free pmp action. If $u \in \mathcal{N}_{\mathrm{L}^\infty(X)\rtimes\Gamma}(\mathrm{L}^\infty(X))$, then there is a partition $X = \bigsqcup A_g$ into measurable subsets and there are S^1-valued elements $a_g \in \mathrm{L}^\infty(A_g) \subset \mathrm{L}^\infty(X)$ such that $u = \sum_g u_g a_g$.*

Proof. We write $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Let $u \in \mathcal{N}_M(A)$. Denote by $\mathrm{E} : M \to A$ the natural conditional expectation and define $A_g := \operatorname{supp} \mathrm{E}(u_g^* u)$ and $a_g = \mathrm{E}(u_g^* u)$. Using Fourier coefficients (Proposition 1.4.23), we see that $u = \sum_g u_g a_g$. We then obtain

$$1 = uu^* = \sum_{g,g'} u_g a_g a_{g'}^* u_{g'}^* = \sum_{g,g'} {}^{g}(a_g a_{g'}^*) u_{gg'^{-1}} .$$

By uniqueness of the Fourier coefficients, we conclude that ${}^{g}(a_g a_{g'}^*) = 0$ if $g \neq g'$ and $\sum_g {}^{g}(a_g a_g^*) = 1$. So $X = \bigsqcup g A_g$ and $a_g \in \mathrm{L}^\infty(A_g)$ is unitary, hence S^1-valued. Further,

$$1 = u^* u = \sum_{g,g'} a_g^* u_g^* u_{g'} a_{g'} = \sum_{g,g'} a_g^* ({}^{g^{-1}g'} a_{g'}) u_{g^{-1}g'} .$$

Comparing Fourier coefficients, we see that $\sum_g a_g^* a_g = 1$, implying that $X = \bigsqcup_g A_g$. This finishes the proof of the proposition. ☐

Theorem 1.4.37 (Singer)**.** *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be free pmp actions of countable groups. Then $(\Gamma \curvearrowright X) \sim_{\mathrm{OE}} (\Lambda \curvearrowright Y)$ if and only if there is a *-isomorphism $\varphi : \mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(Y) \rtimes \Lambda$ such that $\varphi(\mathrm{L}^\infty(X)) = \mathrm{L}^\infty(Y)$.*

Proof. In Theorem 1.4.34 we showed that orbit equivalence implies the existence of a Cartan preserving isomorphism. So it remains to prove the converse implication. Identify $\mathrm{L}^\infty(X) \rtimes \Gamma$ with $\mathrm{L}^\infty(Y) \rtimes Y$ via some Cartan preserving isomorphism. We may then assume that $X = Y$. Denote the natural unitaries of $\mathrm{L}(\Lambda) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ by $(v_h)_h$. Thanks to Theorem 1.4.36, we find for each $h \in \Lambda$ a partition $X = \bigsqcup A_h^g$ and S^1-valued functions $a_h^g \in \mathrm{L}^\infty(A_h^g) \subset \mathrm{L}^\infty(X)$ such that $v_h = \sum_g u_g a_h^g$.

Fix $g \in \Gamma$ and $h \in \Lambda$. If $a \in \mathrm{L}^\infty(X)$ then

$$u_g \mathbb{1}_{A_h^g} a u_g^* = v_h \mathbb{1}_{A_h^g} a v_h^* .$$

So $\mathrm{Ad}\, u_g|_{\mathrm{L}^\infty(A_h^g)} = \mathrm{Ad}\, v_h|_{\mathrm{L}^\infty(A_h^g)}$. Denoting the action of Γ by α and the action of Λ by β, we then conclude that $\alpha_g|_{A_h^g} = \beta_h|_{A_h^g}$ almost everywhere. For fixed $g \in \Gamma$ all but countably many $(A_h^g)_h$ are negligible, since they form a measurable partition of X. It follows that for almost every $x \in X$ there is some $h \in \Lambda$ such that $gx = hx$. So $gx \in \Lambda x$ for almost every $x \in X$. Since Γ is countable, we conclude that $\Gamma x \subset \Lambda x$ for almost every $x \in X$. By symmetry, we obtain $\Gamma x = \Lambda x$ for almost every $x \in X$. □

1.5 Factors

Definition 1.5.1 (Factor)**.** A *factor* is a von Neumann algebra M satisfying $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.

We start by characterising factors as simple von Neumann algebras.

Proposition 1.5.2. *Let M be a von Neumann algebra. Then SOT-closed two-sided ideals in M are precisely of the form zM for some central projection $z \in \mathcal{Z}(M)$.*

In particular, M is a factor if and only M is simple, i.e. M does not contain any SOT-closed two-sided ideals.

Proof. First it is clear that every central projection $z \in \mathcal{Z}(M)$ defines an SOT-closed ideal zM. So let $I \leq M$ be a two-sided SOT-closed ideal and let H be the Hilbert space on which M acts. Let $K := \overline{IH}$. Then I can be considered as a strongly closed *-subalgebra of $\mathcal{B}(K)$, which acts non-degenerately. Since I is a C*-algebra, it contains a bounded approximate unit, so it follows that $\mathrm{id}_K \in I \subset \mathcal{B}(K)$, which is the identity of I. We hence obtain $p_K \in I$, where $p_K : H \to K$ is the orthogonal projection. Further, K is invariant under M' and M, so that $p_K \in M' \cap M = \mathcal{Z}(M)$ by Proposition 1.0.5. So indeed $I = p_K I = p_K M$ for the central projection $p_K \in \mathcal{Z}(M)$.

Now assume that M is a factor and let $I \trianglelefteq M$ be SOT-closed. Then $I = zM$ for some projection $z \in \mathcal{Z}(M) = \mathbb{C}1$. So $I \in \{0, M\}$, showing that M is simple. Assume that M is simple. If $z \in \mathcal{Z}(M)$ is some non-zero projection, then $zM = M$, showing that z is invertible. So $z = 1$. Since $\mathcal{Z}(M) \cong \mathrm{L}^\infty(X)$ for some standard probability measure space X, by Proposition 1.4.35, it follows now that $\mathcal{Z}(M) = \mathbb{C}1$. □

For later use, we observe that every trace on a factor is faithful. As it turn out, there is a unique trace on every finite factor (Theorem 1.5.26)

Lemma 1.5.3. *Let M be a factor and τ a trace on M. Then τ is faithful.*

Proof. Let $I = \{x \in M \mid \tau(x^*x) = 0\}$. The inequality $x^*y^*yx \leq \|y\|^2 x^*x$ shows that I is a left-ideal. Further, $\tau(x^*x) = \tau(xx^*)$, so that I is a two-sided ideal. The characterisation of factors as simple von Neumann algebras from Proposition 1.5.2 now shows that $I = 0$. So τ is faithful. □

Definition 1.5.4. Let M be a finite factor (i.e. a von Neumann algebra which is a factor and finite). If M is infinite dimensional, then it is called a II_1 *factor*.

Remark 1.5.5. The name II_1 factor stems from a classification of factors in different types I_n, $n \in \mathbb{N} \cup \{\infty\}$, II_1, II_∞ and III. Since we focus on finite factors, we will not treat this classification, which can be found for example in Dixmier's book. (and any other book treating von Neumann algebras).

1.5.1 Group factors

The next proposition gives us a first source of II_1 factors.

Proposition 1.5.6. *Let Γ be a discrete group. Then $\mathrm{L}(\Gamma)$ is a factor if and only if Γ is icc, i.e. every non-trivial conjugacy class of Γ is infinite.*

Proof. First assume that there is a finite non-trivial conjugacy class $\{g_1 h g_1^{-1}, \dots, g_n h g_n^{-1}\}$ in Γ. Then $z := \sum_{i=1}^n u_{g_i h g_i^{-1}}$ is a central element of $\mathrm{L}(\Gamma)$. Since δ_e is separating for $\mathrm{L}(\Gamma)$, the fact that $z\delta_e \notin \mathbb{C}\delta_e$ implies that z is not a multiple of the unit. This shows that $\mathrm{L}(\Gamma)$ is not a factor.

Assume that Γ is icc. Take $z \in \mathcal{Z}(\mathrm{L}(G))$ and consider $\hat{z} = z\delta_e \in \ell^2(\Gamma)$. Then $\hat{z} = \widehat{u_g z u_g^*} = u_g z u_g^* \delta_e = u_g z \delta_e u_g^* = u_g \hat{z} u_g^*$, where u_g^* acts on the right of $\ell^2(\Gamma)$ via the right-regular representation. It follows that $\hat{z}$ is a conjugation invariant function in $\ell^2(\Gamma)$. Since every non-trivial conjugacy class of Γ is infinite, 2-sumability implies that $\hat{z}$ is a multiple of δ_e. Since δ_e is a separating vector for $\mathrm{L}(\Gamma)$, this proves that z is a multiple of $1 \in \mathrm{L}(\Gamma)$. So $\mathrm{L}(\Gamma)$ is a factor. □

Exercise 1.5.7. The following groups are icc.

- Non-abelian free groups $\mathbb{F}_n$.
- The group S_∞ of finite permutation of a countable infinite set.

Example 1.5.8. The von Neumann algebras $\mathrm{L}(\mathbb{F}_n)$, $\mathrm{L}(\mathrm{S}_\infty)$ and $\mathrm{L}(\mathbb{F}_n \times \mathrm{S}_\infty)$ are II_1 factors.

1.5.2 Two factoriality criteria for group-measure space constructions

Proposition 1.5.9. *Let $\Gamma \curvearrowright X$ be a pmp action of a discrete group. If $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a factor, then $\Gamma \curvearrowright X$ is ergodic.*

Proof. Assume that $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a factor and take a Γ-invariant measurable subset $A \subset X$. Then $u_g \mathbb{1}_A u_g^* = \mathbb{1}_{gA} = \mathbb{1}_A$ for every $g \in \Gamma$. So $\mathbb{1}_A$ commutes with $\mathrm{L}^\infty(X), \mathrm{L}(G) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$. So $\mathbb{1}_A \in \mathcal{Z}(\mathrm{L}^\infty(X) \rtimes \Gamma) = \mathbb{C}1$, implying that A is either negligible or conegligible. This shows that $\Gamma \curvearrowright X$ is ergodic. □

Proposition 1.5.10. *Let $\Gamma \curvearrowright X$ be a free ergodic pmp action of a discrete group. Then $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a factor.*

Proof. Since $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ is a MASA by Proposition 1.4.28, we have $\mathcal{Z}(\mathrm{L}^\infty(X) \rtimes \Gamma) \subset \mathrm{L}^\infty(X)$. Let $a \in \mathrm{L}^\infty(X)$ be central in the group-measure space construction. Then $a = u_g a u_g^* = {}^g a$ for all $g \in \Gamma$. So a is a Γ-invariant function. Let t be an essential value of a, i.e. $z \in \mathbb{C}$ such that $\{x \in X \mid |a(x) - z| < \varepsilon\}$ is non-negligible for all $\varepsilon > 0$. Since a is Γ-invariant, also $\{x \in X \mid |a(x) - z| < \varepsilon\}$ is Γ-invariant for every $\varepsilon > 0$ and hence it is conegligible by ergodicity of $\Gamma \curvearrowright X$. This shows that a is almost surely equal to z and hence $\mathcal{Z}(\mathrm{L}^\infty(X) \rtimes \Gamma) = \mathbb{C}1$. □

Proposition 1.5.11. *Let $\Gamma \curvearrowright X$ be an ergodic pmp action of an icc discrete group. Then $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a factor.*

Proof. Let $x \in \mathcal{Z}(\mathrm{L}^\infty(X) \rtimes \Gamma)$ and write $x = \sum_{g \in \Gamma} x_g u_g$. For $h \in \Gamma$, we have $x = u_h x u_h^* = \sum_{g \in \Gamma} {}^h x_g u_{hgh^{-1}} = \sum_{g \in \Gamma} {}^h x_{h^{-1}gh} u_g$. Comparing Fourier coefficients, we see that $x_g = {}^h x_{h^{-1}gh}$ for all $g, h \in \Gamma$. Since Γ is icc and $\Gamma \curvearrowright X$ is probability measure preserving, $\| \ \|_2$-summability of $\sum_{g \in \Gamma} x_g u_g$ implies that $x_g = 0$ if $g \neq e$. We showed that $\mathcal{Z}(\mathrm{L}^\infty(X) \rtimes \Gamma) \subset \mathrm{L}^\infty(X)$. Now we can proceed as in the proof of Proposition 1.5.10 and use ergodicity of $\Gamma \curvearrowright X$ to conclude the proof. ☐

Example 1.5.12. If Γ is any infinite group, then non-trivial Bernoulli shifts $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma = (X, \mu)$ are free ergodic and probability measure preserving according to Exercise 1.4.12. Hence the von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a II_1 factor.

1.5.3 Discrete factors and comparison of projections

We already noticed that projections and partial isometries play a crucial role in von Neumann algebras. In this section we take the opportunity to study them in more detail. It seems due to reconsider the polar decomposition, giving rise to an abundance of partial isometries in a von Neumann algebra.

The following notion of order on projections specialises the order on self-adjoint elements of a C*-algebra.

Definition 1.5.13. Let $p, q \in \mathcal{B}(H)$ be projections. We say that $p \leq q$ if $pq = q$.

Note that indeed, $p \leq q$ implies that $q - p$ is a projection and in particular it is positive. We have $(q-p)^*(q-p) = q^2 - qp - pq + p^2 = q - p$. Further, $p \leq q$ if and only if $pH \subset qH$.

Definition 1.5.14. Let $x \in \mathcal{B}(H)$ be an operator. The (geometric) image of x is the subspace $xH \subset H$ and the (geometric) support of x is the subspace $(\ker x)^\perp$. The image (projection) $\operatorname{supp} x$ of x is the smallest projection $p \in \mathcal{B}(H)$ such that $px = x$ and the support (projection) $\operatorname{im} x$ of x is the smallest projection $q \in \mathcal{B}(H)$ such that $xq = x$.

Proposition 1.5.15. *Let M be a von Neumann algebra and $x \in M$. Then $\operatorname{supp} x, \operatorname{im} x \in M$.*

Proof. Since $\operatorname{supp} x = \operatorname{im}(x^*)$, it suffices to shows that $\operatorname{im} x \in M$. Denote by H the Hilbert space on which M acts. The image xH is invariant under M' and so is its closure $\overline{xH}$. So the orthogonal projection p onto $\overline{xH}$ lies in M. We show that $p = \operatorname{im} x$. We have $px\xi = x\xi$ for all $\xi \in H$, so that $px = x$ follows. If $q \in \mathcal{B}(H)$ is another projection satisfying $qx = x$, then $pH = \overline{xH} \subset qH$. So $p \leq q$. This finishes the proof of the proposition. ☐

Proposition 1.5.16 (Polar decomposition)**.** *Let $x \in \mathcal{B}(H)$. Then there is a unique partial isometry $v \in \mathcal{B}(H)$ such that*

- $v|x| = x$
- $v^* v = \operatorname{supp} x$
- $v v^* = \operatorname{im} x$.

If $M \subset \mathcal{B}(H)$ is a von Neumann algebra and $x \in M$, then also $v, |x| \in M$.

Proof. Let $x \in \mathcal{B}(H)$. For every $\xi \in H$ we have

$$\| |x|\xi \| = \langle |x|\xi, |x|\xi \rangle = \langle |x|^2\xi, \xi \rangle = \langle x^*x\xi, \xi \rangle = \langle x\xi, x\xi \rangle = \| x\xi \| .$$

Denoting by $K = \overline{|x|H}$ and $L = \overline{xH}$ the support and the image of x, we see that $v_0 : |x|\xi \mapsto x\xi$ defines a unitary from K to L. Now let v be the partial isometry $v = \iota_L \circ v_0 \circ p_K$, where $\iota_L : L \to H$ denotes the inclusion of L and $p_K : H \to K$ denotes the orthogonal projection onto K. Now $x\xi = v|x|\xi$ for all $\xi \in H$, showing that $x = v|x|$ indeed. By construction we have $v^*v = \operatorname{supp} x$ and $vv^* = \operatorname{im} x$.

Let us prove uniqueness of the decomposition $x = v|x|$. Assume that $x = u|x|$ for some partial isometry satisfying $u^*u = \operatorname{supp} x$ and $uu^* = \operatorname{im} x$. Since $u^*u = \operatorname{supp} x = \operatorname{supp} |x|$, it suffices to check $u|_K = v|_K$. On the dense subspace $|x|H \subset K$, this follows from the identity $u|x| = x$, which implies $u|x|\xi = x\xi = v|x|\xi$ for all $\xi \in H$. Continuity now finishes the proof of uniqueness.

Now assume that x lies in a von Neumann algebra $M \subset \mathcal{B}(H)$ and denote by $x = v|x|$ its polar decomposition. Functional calculus says that $|x| \in M$. Since $\overline{|x|H}$ is a subspace invariant under M', we have $\operatorname{supp} |x| \in M$. Let $u \in \mathcal{U}(M')$. Then uvu^* is a partial isometry satisfying

$$(uvu^*)^*(uvu^*) = uv^*vu^* = u(\operatorname{supp} x)u^* = \operatorname{supp} x ,$$

since $\operatorname{supp} x \in M$ and $u \in M'$. Similarly, we see that $(uvu^*)(uvu^*)^* = \operatorname{im} x$. Moreover,

$$uvu^*|x| = uv|x|u^* = uxu^* = x .$$

Uniqueness of the polar decomposition shows $uvu^* = v$. But this means that $v \in M'' = M$. □

extend to polar decomposition for affiliated operators in L^2.

Definition 1.5.17. Let M be a von Neumann algebra and $p, q \in M$ projections. We say that p dominates q if there is a partial isometry $v \in M$ such that $vv^* \leq p$ and $v^*v = q$. In this case we write $p \succ q$. If $p \prec q$ and $p \succ q$, then p and q are called Murry-von Neumann equivalent and we write $p \sim q$.

The next lemma addresses the subtlety, that à priori equivalent projections are not comparable by a single partial isometry. It holds true for all von Neumann algebras (see Dixmier - Les algèbres d'opérateurs dans l'espace Hilbertien, Proposition 1, p. 216), but we prove it here only for finite von Neumann algebras in which case the proof becomes very short.

Lemma 1.5.18. *Let $p, q \in M$ be equivalent projections in a finite von Neumann algebra M. Then there is a partial isometry $v \in M$ such that $\operatorname{supp} v = p$ and $\operatorname{im} v = q$.*

Proof. By definition of equivalence, there are partial isometries $v, w \in M$ such that $vv^* \leq p$, $v^*v = q$ and $ww^* \leq q$, $w^*w = p$. Let $(\tau_i)_i$ be a faithful family of traces on M. We obtain that

$$\tau_i(p) = \tau_i(w^*w) = \tau_i(ww^*) \leq \tau_i(q) = \tau_i(v^*v) = \tau_i(vv^*) \leq \tau_i(p) ,$$

showing that $\tau_i(p) = \tau_i(q)$ for all i. We obtain $\tau_i(p - vv^*) = 0$, which by faithfulness of $(\tau_i)_i$ implies $p = vv^*$. This finishes the proof. □

The next proposition says that we can compare each pair of projection in a factor, with respect to the partial order $\prec$ from Definition 1.5.17.

Proposition 1.5.19. *Let M be a factor. If $p, q \in M$ are projections, then either $p \prec q$ or $p \succ q$.*

Proof. Since M is a factor, the ideal $\overline{MpM}^{\mathrm{SOT}}$ equals M. In particular, there is some non-zero element $x \in pMq$. Let $x = v|x|$ be the polar decomposition and note that $vv^* = \operatorname{im} x \leq p$ and $v^*v = \operatorname{supp} x \leq q$ lie in M by Proposition 1.5.15.

Let $(v_i)_i$ be a maximal family of partial isometries with pairwise orthogonal images $\operatorname{im} v_i \leq p$ and pairwise orthogonal supports $\operatorname{supp} v_i \leq q$. Then $v = \sum_i v_i$ (in SOT) is a well-defined element of M. If $\operatorname{im} v = \sum_i \operatorname{im} v_i \neq p$ and $\operatorname{supp} v = \sum_i \operatorname{supp} v_i \neq q$, then we can apply the first part of the proposition to $p - \operatorname{im} v$ and $q - \operatorname{supp} v$, so as to obtain a contradiction to the maximality of $(v_i)_i$. This shows that v witnesses either $p \prec q$ or $p \succ q$. □

Next we are going to study von Neumann algebras containing projections that cannot be "split".

Definition 1.5.20. Let M be a von Neumann algebra. A projection $p \in M$ is called *minimal* if $q \leq p$ implies $q \in \{0, p\}$ for any other projection $q \in M$.

Theorem 1.5.21. *Let M be a not necessarily separable factor that contains a minimal projection. Then $M \cong \mathcal{B}(H)$ for some Hilbert space H. In particular, a II_1 factor does not contain any minimal projections.*

Proof. Let $p \in M$ be a minimal projection and $q \in M$ a some non-zero projection. Then $p \prec q$ or $q \prec p$. But $q \prec p$ implies $p \sim q$, since p is minimal. We conclude that $p \prec q$ and hence q contains some minimal projection. This argument shows that (i) every non-zero projection in M contains a minimal projection, and (ii) all minimal projections in M are equivalent.

Let $(p_i)_{i \in I}$ be a maximal family of pairwise orthogonal minimal projections in M. If $\sum_{i \in I} p_i \neq 1$ (where the limit is taken in the strong sense), then there is a minimal projection contained in $1 - \sum_{i \in I} p_i$, which contradicts maximality of $(p_i)_i$. We conclude that $\sum_i p_i = 1$.

Since $p_i \sim p_j$ for all $i, j \in I$, there are partial isometries $v_j^i \in M$ such that $\operatorname{im} v_j^i = p_i$, $\operatorname{supp} v_j^i = p_j$. Note that $p_i M p_i = \mathbb{C} p_i$, by minimality of p_i. We claim that $p_i M p_j = \mathbb{C} v_j^i$. Let $x \in p_i M p_j$ be non-zero. Then $x^*x \in p_j M p_j = \mathbb{C} p_j$ and $xx^* \in p_i M p_i = \mathbb{C} p_i$ are non-zero elements. So if $x = v|x|$ denotes the polar decomposition if x, then $|x| \in \mathbb{C} p_j M p_j$ and v is partial isometry with support p_j and image p_i. We have $v(v_j^i)^* \in p_i M p_i = \mathbb{C} p_i$, so that $v = v p_j = v(v_j^i)^* v_j^i \in \mathbb{C} v_j^i$. We conclude that $x = v|x| \in \mathbb{C} v_j^i p_j = \mathbb{C} v_j^i$, finishing the proof of the claim.

We scaling the partial isometries $(v_j^i)_{i,j \in I}$, we may assume that $v_j^i v_k^j = v_k^i$ for all $i, j, k \in I$. Further $x = \sum_{i,j \in I} p_i x p_j$ for every $x \in M$, allows us to infer that M is generated by the family $(v_j^i)_{i,j \in I}$.

Fix some element $0 \in I$ and write $K := H_0$. A short calculation shows that $W : \ell^2(I) \otimes K \to H$ by $\delta_i \otimes \xi \mapsto v_0^i \xi$ defines a unitary. We prove that $W^* M W = \mathcal{B}(\ell^2(I)) \otimes 1$. For $i, j, j \in I$ and $\xi \in K$ we have

$$W^* v_j^i W(\delta_k \otimes \xi) = W^* v_j^i v_0^k \xi = \delta_{j,k} W^* v_0^i \xi = \delta_{j,k}(\delta_i \otimes \xi)\,.$$

In particular, $W^* M W$ contains all rank one operators in $\mathcal{B}(\ell^2(I)) \otimes 1$, showing that $W^* M W \supset \mathcal{B}(\ell^2(I)) \otimes 1$. It remains to show that $W^* M W \subset (1 \otimes \mathcal{B}(K))' = \mathcal{B}(\ell^2(I)) \otimes 1$. Let $i, j, k \in I$, let $x \in \mathcal{B}(K)$ and $\xi \in K$. Then

$$W^* v_j^i W(1 \otimes x)(\delta_k \otimes \xi) = W^* v_j^i W(\delta_k \otimes x\xi) = \delta_{j,k} \delta_i \otimes x\xi\,.$$

and

$$(1 \otimes x) W^* v_j^i W(\delta_k \otimes \xi) = \delta_{j,k}(1 \otimes x)\delta_i \otimes \xi = \delta_{j,k} \delta_i \otimes x\xi\,.$$

This shows $[W^* v^i_j W, 1 \otimes x] = 0$ for all $i, j \in I$ and all $x \in \mathcal{B}(K)$. Since M is generated by the partial isometries $(v^i_j)_{i,j \in I}$, this shows $M \cong \mathcal{B}(\ell^2(I))$.

Now assume that M is a finite factor containing a minimal projection. Denote by τ some faithful trace on M. Then the family $(p_i)_{i \in I}$ is finite, since $1 = \tau(1) = \sum_{i \in I} \tau(p_i)$ and $\tau(p_i)$ does not depend on i. It follows that $M \cong \mathcal{B}(\ell^2(I)) \cong \mathrm{M}_{|I|}(\mathbb{C})$ is a matrix algebra and hence finite dimensional. ☐

We finish this section by another important proposition, underlining the importance of projections in von Neumann algebras.

Proposition 1.5.22. *Let M be a von Neumann algebra. Then M is the norm closure of the linear span of all its projections.*

Proof. Splitting elements of M in their real and imaginary parts $\frac{1}{2}(x + x^*)$ and $\frac{1}{2i}(x - x^*)$, it suffices to show that every self-adjoint element in M can be approximated in norm by a finite linear combination of projections.

Let $x \in M$ be self-adjoint. Then $\{x\}''$ is an abelian von Neumann algebra isomorphic to $\mathrm{L}^\infty(\sigma(x))$ by Proposition 1.4.35. Here x is identified with the identity function on $\sigma(x)$. Covering $\sigma(x)$ by intervals of length ε, we can approximate $\mathrm{id}_{\sigma(x)}$ in the uniform topology up to ε by a finite linear combination of indicator functions. This finishes the proof. ☐

1.5.4 The unique trace on a II_1 factor

In this section we are going to show that finite factors are characterised as those finite von Neumann algebras which admit exactly one trace. Our first observation is that a finite von Neumann algebra with a unique trace must be a factor.

Proposition 1.5.23. *Let M be a finite von Neumann algebra which admits a unique trace. Then M is a factor.*

Proof. Let τ be the unique trace on M, which must be faithful by the finiteness assumption on M. Consider a projection $p \in \mathcal{Z}(M)$ satisfying $\tau(p) \neq 0$. Then $\tau_p(x) = \tau(pxp)$ defines a trace on M, which must be a positive multiple of τ by uniqueness. Now $\tau_p(1-p) = 0$ implies $\tau(1-p) = 0$. This shows that $p = 1$, since τ is faithful. So $\mathcal{Z}(M)$ contains only the projections 0 and 1 and thus $\mathcal{Z}(M) = \mathbb{C}1$. ☐

In the rest of this section we are going to prove the converse to Proposition 1.5.23. We start by quantifying the absence of minimal projections in II_1 factors (compare with Theorem 1.5.21).

Lemma 1.5.24. *Let M be a II_1 factor with trace τ. Then for every $\varepsilon > 0$ there is a non-zero projection $p \in M$ such that $\tau(p) < \varepsilon$.*

Proof. Assume that there is $\varepsilon > 0$ such that every non-zero projection $p \in M$ satisfies $\tau(\varepsilon) \geq \varepsilon$. We will show that M contains a minimal projection. Then Theorem 1.5.21 shows that M is a matrix algebra and hence finite dimensional.

If $p \in M$ is an arbitrary projection, then either p is minimal or there is a properly contained non-zero projection $p' < p$. We have $\tau(p - p') \leq \tau(p) - \varepsilon$. We conclude that M either contains a minimal projection or a non-zero projection p such that $\tau(p) < 2\varepsilon$. However the latter condition already implies that p is minimal, by our assumptions. This finishes the proof. ☐

Lemma 1.5.25. *Let M be a II_1 factor. Then there is a sequence of projections $(p_i)_{i\in\mathbb{N}}$ in M such that for every $i \in \mathbb{N}$ there are partial isometries $v_i^j \in M$, $j \in \{1, \dots, 2^i\}$ such that $(v_i^j)^* v_i^j = p_i$ and $\sum_{j=1}^{2^i} v_i^j (v_i^j)^* = 1$.*

A sequence of projections as described in the last lemma is called a *fundamental sequence of projections*.

Proof. It suffices to show that for every projection $p \in M$ there are projections $p_1, p_2 \in M$ such that $p_1 \sim p_2$ and $p = p_1 + p_2$. Considering the von Neumann algebra pMp instead of M, we may assume that $p = 1$.

Let τ be a trace on M. Let $(q_i)_{i\in I}$ be a maximal family of pairwise orthogonal projections such that $\tau(\sum_{i\in I} q_i) \leq 1/2$. We show that $q := \sum_{i\in I} q_i$ satisfies $c := \tau(q) = 1/2$. If this was not the case, Lemma 1.5.24 applied to $(1-q)M(1-q)$ gives us a non-zero projection $q' \leq 1 - q$ such that $\tau(q') < 1/2 - c$. So we can add q' to the family $(q_i)_{i\in I}$ and contradict its maximality. This finishes the proof. □

Theorem 1.5.26. *A finite von Neumann algebra M is a factor if and only if it admits a unique trace.*

Proof. If M admits a unique trace, then M is a factor by Proposition 1.5.23. So we have to show that there is a unique trace on every finite factor M.

If M is a finite factor, then either $M \cong \mathrm{M}_n(\mathbb{C})$ is a matrix algebra or M is a II_1 factor by Theorem 1.5.21. Since matrix algebras have a unique trace, we may assume that M is a II_1 factor.

By Lemma 1.5.25, we may take a fundamental sequence $(p_i)_{i\in\mathbb{N}}$ for M. Every trace τ on M satisfies

$$1 = \tau(1) = \sum_j \tau(v_i^j (v_i^j)^*) = \sum_j \tau((v_i^j)^* v_i^j) = 2^i \tau(p_i) .$$

Hence $\tau(p_i) = 2^{-i}$. It follows that all traces agree on the SOT-closure of $E = \operatorname{span}\{vv^* \mid v^*v = p_i \text{ for some } i\}$. We show that this set contains all projections of M and hence, by Proposition 1.5.22, all of M.

Fix a trace τ on M. By Proposition 1.5.3, we know that τ is faithful. Let $p \in M$ be some projection and $c := \tau(p)$. Let $i \in \mathbb{N}$ be minimal such that $2^{-i} \leq c$. By Proposition 1.5.19, we have either $p_i \prec p$ or $p_i \succ p$. Since the latter implies $2^{-i} = \tau(p_i) \geq \tau(p) = c \geq 2^{-i}$, we can conclude $p_i \sim p$ in this case. So we obtain $p_i \prec p$ in any case. Let $v \in M$ be a partial isometry satisfying $\operatorname{supp} v = p_i$ and $\operatorname{im} v \leq p$. Then $\tau(p - vv^*) = c - 2^{-i} < 2^{-(i+1)}$ by the choice of i. Inductively, we can approximate p in $\| \cdot \|_{\tau,2}$ by a bounded sequence of elements from E. Since the $\| \ \|_2$-topology agrees with the strong topology on bounded sets by Proposition 1.2.4, we conclude that p lies in the SOT-close of E. This finishes the proof. □

1.6 Solutions to exercises

Solution (To exercise 1.2.6). Consider the von Neumann algebra $\mathrm{L}^\infty([0,1]) \subset \mathrm{L}^\infty([0,1],\lambda)$. We give an example of an unbounded sequence of function $f_n \in \mathrm{L}^\infty([0,1])$ such that $f_n \to 0$ in $\|\ \|_2$, but $f_n \not\to 0$ in the strong topology (not even weakly).

Let

$$f_n(x) := \mathbb{1}_{[2^{-(n+1)},2^{-n}]}\frac{1}{x^{1/2}}$$

and

$$\xi(x) := \frac{1}{x^{1/4}}\,.$$

Note that ξ is well defined up to measure 0 in $\mathrm{L}^2([0,1],\lambda)$. We have

$$\|f_n\|_2 = \int_{2^{-(n+1)}}^{2^{-n}} \frac{1}{x^{1/2}}\mathrm{d}x = \left[x^{1/2}\right]_{2^{-(n+1)}}^{2^{-n}} = 2^{-n/2} - 2^{\frac{-(n+1)}{2}} \to 0\,,$$

while

$$\langle f_n\xi,\xi\rangle = \int_{2^{-(n+1)}}^{2^{-n}} \frac{1}{x}\mathrm{d}x = \left[\log x\right]_{2^{-(n+1)}}^{2^{-n}} = \log(2^{-n}) - \log(2^{-(n+1)}) = \log(2^{-1})\,.$$

So $(f_n)_n$ is the desired example.

Solution (To exercise 1.4.4). Let $\Lambda \trianglelefteq \Gamma$ be a normal inclusion of groups. We show that for every trace τ on $\mathrm{L}(\Lambda)$ the composition $\tau\circ\mathrm{E}$ is a trace on $\mathrm{L}(\Gamma)$.

For $g,h\in\Gamma$, we have $\mathrm{E}(u_gu_h) = \mathbb{1}_\Lambda(gh)u_{gh}$ and $\mathrm{E}(u_hu_g) = \mathbb{1}_\Lambda(hg)u_{hg}$. Since $\Lambda \trianglelefteq \Gamma$ is normal, we have $gh = h^{-1}(hg)h \in \Lambda$ if and only if $hg\in\Lambda$. So if τ denotes any trace on $\mathrm{L}(\Lambda)$, then

$$\tau\circ\mathrm{E}(u_gu_h) = \mathbb{1}_\Lambda(gh)\tau(u_{gh}) = \mathbb{1}_\Lambda(hg)\tau(u_{gh}) = \mathbb{1}_\Lambda(gh)\tau(u_{hg})\,.$$

Solution (To exercise 1.4.12). Let Γ be an infinite group and (X_0,μ_0) a non-trivial standard probability measure space. Denote by $\Gamma \curvearrowright (X,\mu) = (X_0,\mu_0)^\Gamma$ the Bernoulli shift. We show that this is a free ergodic pmp action.

We start by showing that $\Gamma\curvearrowright X$ is probability measure preserving. To this end define for any sequence of pairwise different elements $g_1,\dots,g_n\in\Gamma$ and measurable subsets $A_1,\dots,A_n\subset X_0$ the set $B^{g_1,\dots,g_n}_{A_1,\dots,A_n} = \{x\in X \mid \forall i\in\{1,\dots,n\} : x_{g_i}\in A_i\}$. Then $\mu(B^{g_1,\dots,g_n}_{A_1,\dots,A_n}) = \prod_{i=1}^n \mu_0(A_i)$ by the definition of the product measure. Further, if $g\in\Gamma$, then $gB^{g_1,\dots,g_n}_{A_1,\dots,A_n} = B^{gg_1,\dots,gg_n}_{A_1,\dots,A_n}$. So $mu(gB^{g_1,\dots,g_n}_{A_1,\dots,A_n}) = \mu(B^{g_1,\dots,g_n}_{A_1,\dots,A_n})$. Since subsets of the form $B^{g_1,\dots,g_n}_{A_1,\dots,A_n}$ generate the σ-algebra of X, it follows indeed that μ is Γ-invariant.

Let us next show that $\Gamma\curvearrowright X$ is free. Let $g\in\Gamma\setminus\{e\}$. Then $gx = x$ implies that x is constant on $\langle g\rangle$-orbits. So

$$X_g = \{x\in X \mid gx = x\} = \Delta(X_0^{\Gamma/\langle g\rangle})\,,$$

where Δ denotes the diagonal embedding $\Delta(x)_h = (x)_{h\langle g\rangle}$. Assuming that g has infinite order, we can use the fact that the diagonal embedding $X_0\hookrightarrow X_0^{\mathbb{N}}$ has measure 0 to conclude that $\mu(X_g) = 0$. If $\mathrm{ord}(g)<\infty$, then $\Gamma/\langle g\rangle$ is infinite. The image of the embedding $X_0\hookrightarrow X^{\mathrm{ord}(g)}$ has measure $m<1$. So $\Delta(X_0^{\Gamma/\langle g\rangle})$ has measure $\prod_{h\langle g\rangle\in\Gamma/\langle g\rangle} m = 0$.

We finally show that $\Gamma \curvearrowright X$ is ergodic. To this end we consider the following property. The action $\Gamma \curvearrowright X$ is mixing if for all $A, B \subset X$ measurable we have $\mu(A \cap gB) \overset{g\to\infty}{\to} \mu(A)\mu(B)$. A mixing action of an infinite group is ergodic, since a Γ-invariant set $A \subset X$ satisfies

$$\mu(A) = \mu(A \cap gA) \to \mu(A)^2$$

implying that $\mu(A) \in \{0,1\}$. So proving that the Bernoulli shift is mixing, shows that it is ergodic. Let $g_1, \dots, g_n$ and $h_1, \dots, h_m$ be two finite sequences of pairwise different elements from Γ and let $A_1, \dots, A_n$ and $B_1, \dots, B_m$ be measurable subsets of X_0. Let $F = \{g \in \Gamma \mid \exists i \in \{1, \dots, n\} \exists j \in \{1, \cdots, m\} : gh_j = g_j\}$, which is a finite subset of Γ. If $g \in \Gamma \setminus F$, then $B^{g_1,\dots,g_n}_{A_1,\dots,A_n} \cap gB^{h_1,\dots,h_m}_{B_1,\dots,B_m} = B^{g_1,\dots,g_n,gh_1,\dots,gh_m}_{A_1,\dots,A_n,B_1,\dots,B_m}$. In particular,

$$\mu(B^{g_1,\dots,g_n}_{A_1,\dots,A_n} \cap gB^{h_1,\dots,h_m}_{B_1,\dots,B_m}) = \mu(B^{g_1,\dots,g_n,gh_1,\dots,gh_m}_{A_1,\dots,A_n,B_1,\dots,B_m}) = \mu(B^{g_1,\dots,g_n}_{A_1,\dots,A_n})\mu(B^{h_1,\dots,h_m}_{B_1,\dots,B_m})\,.$$

If now A, B are arbitrary measurable subsets of X, they can be approximated by finite disjoint unions of sets of the form $B^{g_1,\dots,g_n}_{A_1,\dots,A_n}$. This way one proves that for every $\varepsilon > 0$ there is a finite set $F \subset \Gamma$ such that for all $g \in \Gamma \setminus F$ we have

$$|\mu(A \cap gB) - \mu(A)\mu(B)| < \varepsilon\,.$$

This is the exact meaning of $\mu(A \cap gB) \overset{g\to\infty}{\to} \mu(A)\mu(B)$.

Solution (**To exercise 1.4.14**)**.** Let $\Gamma \leq G$ be a dense subgroup of a compact second countable group. Denote by μ the normalised Haar measure of G. We show that the action $\Gamma \curvearrowright (G, \mu)$ is free ergodic and pmp.

Since the Haar measure is left invariant, it is clear that $\Gamma \curvearrowright G$ is pmp. Further, if $g \in \Gamma$ and $x \in G$ satisfy $gx = x$, then $g = e$ just by the fact that Γ is a subgroup of G. It remains to show ergodicity of $\Gamma \curvearrowright G$. Assume that $A \subset G$ is a non-negligible measurable Γ-invariant subset. Consider the probability measure $\nu = \frac{1}{\mu(A)}\mu|_A$ on G. Since A is Γ-invariant, also ν is Γ-invariant. By continuity of the action $G \curvearrowright \mathcal{P}(G)$ on the compact set of all probability measures on G, it follows that ν is also G-invariant. But then uniqueness of the normalised Haar measure (see for example Theorem 1.3.4 in Deitmar, Echerhoff - Principles of Harmonic Analysis) implies that $\mu(A^c) = \nu(A^c) = 0$. So A is co-negligible.

Solution (**To exercise 1.5.7**)**.** We first show that $\mathbb{F}_n$ is icc for all $n \geq 2$. Denote by $x_1, \dots, x_n$ the free letters. Let $g \in \mathbb{F}_n$ be non-trivial and let x_i, $i \in \{1, \dots, n\}$ be the first letter from the left in the reduced form of g. Let $j \in \{1, \dots, n\} \setminus \{j\}$. Then the reduced form of $x_j^l g x_j^{-l}$ starts with $x_j^l x_i$, making these elements pairwise different for $l \in \mathbb{Z}$. It follows that the conjugacy class of g is infinite.

Now consider the group S_∞ of finitely supported permutations of $\mathbb{N}$ and let $\pi \in \mathrm{S}_\infty$ be non-trivial. There is $n \in \mathbb{N}$ such that $\pi(n) \neq n$. We have

$$\big((nm)\pi(nm)\big)(m) = \pi(n)$$

for all $m \neq \pi(n)$. So the elements $(nm)\pi(nm)$ are pairwise different and the conjugacy class of π is infinite.

2 Jones' basic construction and Popa's intertwining technique

Intertwining by bimodules is a techniques introduced by Popa in the mid 2000's in order to study unitary conjugacy of inclusions of von Neumann algebras. The aim of this section is to prove the following equivalence, which gives a practical criterion for conjugation of Cartan subalgebras.

Theorem 2.0.1. *Let $A, B \subset M$ be von Neumann subalgebras of a tracial von Neumann algebra. Then the following two statements are equivalent.*

- *There is a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$, a non-zero *-homomorphism $\varphi : A \to p\big(\mathrm{M}_n(\mathbb{C}) \otimes B\big)p$ and a non-zero partial isometry $v \in \big(\mathrm{M}_{1,n}(\mathbb{C}) \otimes M\big)p$ such that $v\varphi(a) = av$ for all $a \in A$.*
- *There is no sequence of unitaries $(u_n)_n$ in A such that for all $a, b \in M$ we have $\|\mathrm{E}_B(au_nb)\|_2 \to 0$*

If M is a finite factor and $A, B \subset M$ are Cartan subalgebras, then either of the previous conditions implies that there is a unitary $u \in M$ such that $uAu^* = B$.

2.1 Unbounded traces and semi-finite factors

In the sequel we will need to a replacement for traces on von Neumann algebras of the form $N \overline{\otimes} \mathcal{B}(H)$ for a finite von Neumann algebra N and an infinite dimensional Hilbert space H. Such a von Neumann algebra is not finite and in fact it does not support a single non-zero trace. We have to replace traces by their unbounded cousins, like in the passage from $\mathrm{M}_n(\mathbb{C})$ to $\mathcal{B}(\ell^2(\mathbb{N}))$.

Definition 2.1.1. Let M be a von Neumann algebra. An unbounded trace on M is a map $\mathrm{Tr} : M^+ \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$ for all $x \in M$. The trace Tr is called

- semifinite, if the set $\mathfrak{n}_{\mathrm{Tr}} = \{x \in M^+ \mid \mathrm{Tr}(x^*x) < \infty\}$ is *-strongly dense in M;
- faithful, if $\mathrm{Tr}(x^*x) = 0$ implies $x = 0$ for $x \in M$;
- normal, $\mathrm{Tr}(\sup x_n) = \sup \mathrm{Tr}(x_n)$ for all non-decreasing bounded sequences $(x_n)_n$ in M^+.

Note that every tracial state defines a unique normal unbounded trace in the sense of the previous definition.

Definition 2.1.2. A von Neumann algebra M is called semifinite if it admits a faithful family $(\mathrm{Tr}_i)_i$ of normal semifinite traces. A semifinite factor that is not finite or contains a minimal projection is called a II_∞ factor.

The terminology of the previous definition is explained by the next theorem, expressing type II_∞ factors in terms of "infinite amplifications" of II_1 factors. It can be proved using a similar strategy as for Theorem 1.5.21.

Theorem 2.1.3. *If N is a II_1 factor and H and infinite dimensional Hilbert space, then $N \overline{\otimes} \mathcal{B}(H)$ is a type II_∞ factor. Vice versa, if M is a type II_∞ factor, then $M \cong N \overline{\otimes} \mathcal{B}(H)$ for some II_1 factor N and some infinite dimensional Hilbert space H.*

Let us collect some remarks, drawing parallels with theorems we already know for tracial states.

Remark 2.1.4. If M is a von Neumann algebra with an unbounded trace Tr, then $\mathfrak{n}_{\mathrm{Tr}}$ is a two-sided ideal in M, which is not necessarily SOT-closed. In particular, an unbounded trace on a factor is automatically semifinite. In analogy with Theorem 1.5.26, one can proof that a type II_∞ factor admits a unique non-zero unbounded trace up to scaling by positive constants.

add GNS-construction for unbounded traces

In Proposition 1.2.4 we described the equivalence of SOT and of $\| \ \|_2$ convergence on the unit ball of a tracial von Neumann algebra. The reader might have observed, that not only the topologies, but also the uniformities underlying the SOT and the $\| \ \|_2$-topology agree. A similar result holds for von Neumann algebras admitting a faithful semifinite normal trace. We omit its proof here.

Proposition 2.1.5. *Let M be a von Neumann algebra with normal semifinite faithful trace* Tr*. If $(x_i)_i$ is a bounded net in $\mathfrak{n}_{\mathrm{Tr}}$ converging to $\xi \in \mathrm{L}^2(M, \mathrm{Tr})$ in $\| \ \|_{2,\mathrm{Tr}}$, then there is $x \in \mathfrak{n}_{\mathrm{Tr}}$ such that $x_i \to x$ in SOT and $\hat{x} = \xi$.*

Proof. Denote by $\xi \in \mathrm{L}^2(M, \mathrm{Tr})$ the $\| \ \|_{2,\mathrm{Tr}}$-limit of the bounded net $(x_i)_i$ in $\mathfrak{n}_{\mathrm{Tr}}$. Let $c \geq 0$ such that $\|x_i\| \leq c$ for all i. For $y \in \mathfrak{n}_{\mathrm{Tr}}$ we have

$$\|\xi y\| = \lim_i \|\hat{x}_i y\| = \lim_i \|x_i \hat{y}\| \leq c\|y\|_{2,\mathrm{Tr}} .$$

So $\hat{y} \mapsto \xi y$ extends to a bounded operator $\mathrm{L}_\xi : \mathrm{L}^2(M, \mathrm{Tr}) \to \mathrm{L}^2(M, \mathrm{Tr})$. Since $(x_i)_i$ is a bounded net, the previous calculation shows that $x_i \to \mathrm{L}_\xi$ in SOT. In particular, $\mathrm{L}_\xi \in M$.

We show that $\mathrm{L}_\xi \in \mathfrak{n}_{\mathrm{Tr}}$. Note that $\mathrm{L}_\xi \mathrm{L}_\xi^* = \sup_{a \in \mathfrak{n}_{\mathrm{Tr}}, \|a\| \leq 1} \mathrm{L}_\xi a a^* \mathrm{L}_\xi^*$. We have $\mathrm{L}_\xi a \in \mathfrak{n}_{\mathrm{Tr}}$ and $\widehat{\mathrm{L}_\xi a} = \mathrm{L}_\xi \hat{a} = \xi a$ by the definition of L_ξ. This shows that

$$\mathrm{Tr}(\mathrm{L}_\xi^* \mathrm{L}_\xi) = \mathrm{Tr}(\mathrm{L}_\xi \mathrm{L}_\xi^*) = \sup_{a \in \mathfrak{n}_{\mathrm{Tr}}, \|a\| \leq 1} \mathrm{Tr}(\mathrm{L}_\xi a a^* \mathrm{L}_\xi^*) = \sup_{a \in \mathfrak{n}_{\mathrm{Tr}}, \|a\| \leq 1} \|\xi a\|_{2,\mathrm{Tr}}^2 \leq \sup_{a \in \mathfrak{n}_{\mathrm{Tr}}, \|a\| \leq 1} \|\xi\|^2 \|a\|^2 \leq \|\xi\| .$$

So $\mathrm{L}_\xi \in \mathfrak{n}_{\mathrm{Tr}}$. The equation $\hat{\mathrm{L}}_\xi a = \xi a$ for all $a \in \mathfrak{n}_{\mathrm{Tr}}$ implies that $\hat{\mathrm{L}}_\xi = \xi$. This finishes the proof of the proposition. ☐

Here's an auxiliary result that we use frequently in L^2-space of von Neumann algebras. Before we show its proof, recall the parallelogram identity

$$\frac{1}{2}\left(\|\xi + \eta\|^2 + \|\xi - \eta\|^2\right) = \|\xi\|^2 + \|\eta\|^2 ,$$

which holds in $\mathbb{C}^2$ and hence in every Hilbert space.

Proposition 2.1.6. *Let H be a Hilbert space and $C \subset H$ a closed convex set. Then there is a unique element of minimal norm in C.*

Proof. Let $\ell = \inf_{\xi \in C} \|\xi\|$ and take a sequence $(\xi_n)_n$ in H such that $\|x_n\| \to \ell$ as $n \to \infty$. The fact that C is convex together with the parallelogram identity implies that

$$\ell^2 \leq \|\frac{1}{2}(\xi_n + \xi_m)\|^2 = \frac{1}{2}(\|\xi_n\|^2 + \|\xi_m\|^2) - \frac{1}{4}\|\xi_n - \xi_m\|^2 .$$

As $\frac{1}{2}(\|\xi_n\|^2 + \|\xi_m\|^2)$ converges to ℓ^2 when $n, m \to \infty$, we conclude that $(\xi_n)_n$ is a Cauchy sequence. Denote by $\xi \in C$ the limit of $(\xi_n)_n$, which satisfies $\|\xi\| = \ell$.

If $\xi, \eta \in C$ satisfy $\|\xi\| = \|\eta\| = \ell$, then

$$\ell^2 \leq \|\frac{1}{2}(\xi_n + \xi_m)\|^2 = \ell^2 - \frac{1}{4}\|\xi - \eta\|^2 \leq \ell^2$$

shows that $\xi = \eta$. This finishes the proof of the proposition. ☐

2.2 The J-operator

Proposition 2.2.1. *Let M be a von Neumann algebra with normal semifinite faithful trace* Tr*. The map $\mathfrak{n}_{\mathrm{Tr}} \ni x \mapsto x^*$ extends to a conjugate linear isometric involution J of $\mathrm{L}^2(M, \mathrm{Tr})$.*

Proof. From the tracial property $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$, it follows that J is $\| \ \|_{2,\mathrm{Tr}}$-preserving. So it extends form the dense subset $\mathfrak{n}_{\mathrm{Tr}} \subset \mathrm{L}^2(M, \mathrm{Tr})$ to a well-defined conjugate linear map $J : \mathrm{L}^2(M, \mathrm{Tr}) \to \mathrm{L}^2(M, \mathrm{Tr})$. Since $(x^*)^* = x$ for all $x \in M$, we have $J^2 = \mathrm{id}$, so J is an isometric involution of $\mathrm{L}^2(M, \mathrm{Tr})$. □

Theorem 2.2.2. *Let M be a von Neumann algebra with normal semifinite faithful trace* Tr*. The J operator satisfies $JMJ = M' \subset \mathcal{B}(\mathrm{L}^2(M, \mathrm{Tr}))$. Further, $x^{\mathrm{op}} \mapsto Jx^*J$ defines an isomorphism $M^{\mathrm{op}} \cong M'$.*

2.3 Modules over von Neumann algebras

Definition 2.3.1. Let M be a von Neumann algebra. A left M-module is a Hilbert space $\mathcal{H}$ with a normal *-homomorphism $M \to \mathcal{B}(\mathcal{H})$. We write ${}_M\mathcal{H}$ to indicate that $\mathcal{H}$ is a left M-module. A right M-module is a Hilbert space $\mathcal{H}$ with a normal *-antihomomorphism $M \to \mathcal{B}(\mathcal{H})$, that is a normal $\mathbb{C}$-linear antimultiplicative map which respects the *-structure. We write $\mathcal{H}_M$ for a right M-module.

Example 2.3.2. If M is a von Neumann algebra with faithful semifinite normal trace, then $\mathrm{L}^2(M, \mathrm{Tr})$ is a left M-module and a right M-module. Further, $\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)$ is a both a left and a right M-module and if $p \in \mathcal{B}(\ell^2(\mathbb{N})) \otimes M$ is a projection, then also ${}_M\big(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)\big)p$ is a left M-module and $p\big(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)\big)_M$ is a right M-module.

Theorem 2.3.3. *Let M be a tracial von Neumann algebra. If ${}_M\mathcal{H}$ is a countably generated left M-module, then there is a projection $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$ such that ${}_M\mathcal{H} \cong \big(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)\big)p$. If $\mathcal{H}_M$ is a countably generated right M-module, then there is a projection $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$ such that $\mathcal{H}_M \cong p\big(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)\big)_M$.*

In order to prove this theorem, we will make use of the existence of bounded vectors in any M-module, that is denoting by ${}_M\mathcal{H}$ a left M-module, then there is a dense subspace ${}^0\mathcal{H} \leq \mathcal{H}$ such that $\mathrm{R}_\xi : M \to \mathcal{H} :\ni x \mapsto x\xi$ extends to a bounded map $\mathrm{L}^2(M) \to \mathcal{H}$. (Here R_ξ stands for "right multiplication with ξ"). A proof of this fact lies unfortunately beyond the scope of this course.

Proof of Theorem 2.3.3. Let us first reduce to the case of left M-modules. If $\mathcal{H}_M$ is a right module, then $x\overline{\xi} = \overline{\xi x^*}$ defines a left M-module structure on the conjugate Hilbert space $\overline{\mathcal{H}}$. In a similar way we can pass from left to right M-modules, establishing a bijection between the set of countably generated left M-modules and countably generated right M-modules. In particular, the $\overline{{}_M\mathrm{L}^2(M)} \cong \mathrm{L}^2(M)_M$ and more generally $\overline{{}_M\big(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)\big)p} \cong p\big(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)\big)_M$.

Let ${}_M\mathcal{H}$ be a countably generated left M-module. Let $(V_n)_n$ be a maximal family of non-zero M-linear partial isometries $V_n : \mathrm{L}^2(M) \to \mathcal{H}$ with pairwise orthogonal images $V_n\mathrm{L}^2(M) = \mathcal{K}_n$. If $\bigoplus_n \mathcal{K}_n \neq \mathcal{H}$, then we can take a bounded vector ξ in its orthogonal complement $\mathcal{K}$, which is an M-submodule of $\mathcal{H}$. Denote by V the partial isometry in the polar decomposition of the bounded operator $\mathrm{R}_\xi : \mathrm{L}^2(M) \to \mathcal{K}$. Then $V\mathrm{L}^2(M) \leq \mathcal{K}$ is orthogonal to all $(\mathcal{K}_n)_n$ and V is M-linear, since R_ξ is M-linear. This contradicts maximality of the family $(V_n)_n$ hand hence shows that $\mathcal{H} = \bigoplus \mathcal{K}_n$. Since $\mathcal{H}$ is countably generated, the family of its pairwise orthogonal non-zero subspaces $(\mathcal{K}_n)_n$ is countable. If there it is a finite family, we may simply drop the non-zero assumption on all $(V_n)_n$ and complete it to a countably infinite family. We identify this family's index set with $\mathbb{N}$. Define a map $W = \bigoplus_{n\in\mathbb{N}} V_n : \ell^2(\mathbb{N})\otimes \mathrm{L}^2(M) \to \mathcal{H}$, which is an M-linear coisometry. Since W is M-linear, its support projection lies in $(1\otimes M)' \cap \big(\mathcal{B}(\ell^2(\mathbb{N}))\otimes \mathrm{L}^2(M)\big)$. Using the fact that $M' \cap \mathcal{B}(\mathrm{L}^2(M))$ is antiisomorphic with M, we obtain a projection $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes} M$ and an M-linear unitary between $\big(\ell^2(\mathbb{N})\otimes \mathrm{L}^2(M)\big)p \to \mathcal{H}$. This finishes the proof of the theorem. □

2.4 Jones' basic construction

One important technical tool in von Neumann algebras is Jones' basic construction, which implements a conditional expectation $\mathrm{E} : M \to B$ as conjugation by a projection $e_B \in \mathrm{L}^2(M)$ satisfying $e_B x e_b = \mathrm{E}(x) e_B$. As for the group-measure space constructions, we proceed in an axiomatic way.

Definition 2.4.1. Let $B \subset M$ be an inclusion of tracial von Neumann algebras and $\mathrm{E} : M \to B$ the trace preserving conditional expectation. Then the basic construction of $B \subset M$ is a von Neumann algebra $\langle M, e_B\rangle$ such that

- $M \subset \langle M, e_B\rangle$
- there is a projection $e_B \in \langle M, e_B\rangle$,
- $\langle M, e_B\rangle$ is generated by M and e_B,
- $e_B x e_B = \mathrm{E}(x) e_B$ for all $x \in M$, and
- there is a faithful normal (unbounded) trace Tr on $\langle M, e_B\rangle$ satisfying $\mathrm{Tr}(x e_B x^*) = \tau(x^* x)$ for all $x \in M$.

Theorem 2.4.2. *The basic construction exists and it is unique up to isomorphism preserving M and e_B and the unbounded trace.*

Proof. Consider $M \subset \mathcal{B}(\mathrm{L}^2(M))$ and let $e_B : \mathrm{L}^2(M) \to \mathrm{L}^2(B)$ be the orthogonal projection. We show that $\mathcal{M} := (M \cup \{e_B\})''$ is the basic construction. By definition $M \subset \mathcal{M}$ and $e_B \in \mathcal{M}$. Further $\mathcal{M}$ is generated by M and e_B. For $x \in M$, the element $e_B \hat{y} \in \mathrm{L}^2(B)$ is uniquely determined by its scalar products with $\hat{y}$, for $y \in \mathrm{L}^2(B)$. We obtain for $x \in M$, $y \in B$

$$\langle e_B \hat{x}, \hat{y}\rangle = \langle \hat{x}, \hat{y}\rangle = \tau(y^* x) = \tau(\mathrm{E}_B(y^* x)) = \tau(y^*, \mathrm{E}_B(x)) = \langle \widehat{\mathrm{E}_B(x)}, \hat{y}\rangle$$

This shows that $e_B \hat{x} = \widehat{\mathrm{E}_B(x)}$ for all $x \in M$.

If $x, y \in M$, then

$$e_B x e_B \hat{y} = e_B x \widehat{\mathrm{E}_B(y)} = \big(\mathrm{E}_B(x \mathrm{E}_B(y))\big)^\wedge = \big(\mathrm{E}_B(x)\mathrm{E}_B(y)\big)^\wedge = \mathrm{E}_B(x) e_B \hat{y}$$

So $e_B x e_B = \mathrm{E}_B(x)$ for all $x \in M$.

It remains to show that there is a faithful normal trace Tr on $\mathcal{M}$ satisfying $\mathrm{Tr}(x e_B x^*) = \tau(x^* x)$ for all $x \in M$. To this end not that $\mathcal{M}' = JMJ \cap \{e_B\}' = JBJ$. By Theorem 2.3.3, we find an isomorphism $U : \mathrm{L}^2(M)_B \to p\big(\ell^2(\mathbb{N})\otimes \mathrm{L}^2(B)\big)_B$. We may assume that $U(\hat{1}) = \delta_1 \otimes \hat{1} \in \ell^2(\mathbb{N}) \otimes \mathrm{L}^2(B)$ and hence $p \geq e_1 \otimes 1$. Note that $U * e_B U^* = e_1 \otimes 1$. The faithful normal trace $\mathrm{Tr}_{\ell^2(\mathbb{N})} \otimes \tau_B$ on $\mathcal{B}(\ell^2(\mathbb{N})) \otimes B =$

$\mathcal{B}(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(B)) \cap JBJ'$ restricts to $p\big(\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}B\big)p = U\mathcal{M}U^*$. Define $\mathrm{Tr} = (\mathrm{Tr}_{\ell^2(\mathbb{N})} \otimes \tau_B) \circ (\mathrm{Ad}\, U)$. Then

$$\begin{aligned}\mathrm{Tr}(xe_Bx^*) &= \mathrm{Tr}(e_Bx^*xe_B)\\ &= \mathrm{Tr}(\mathrm{E}_B(x^*x)e_B)\\ &= (\mathrm{Tr}_{\ell^2(\mathbb{N})} \otimes \tau_B)(e_1 \otimes \mathrm{E}_B(x^*x))\\ &= \tau_B(\mathrm{E}_B(x^*x))\\ &= \tau(x^*x)\,.\end{aligned}$$

This proves existence of the basic construction.

We only sketch a proof of the uniqueness statement in the Theorem. Let $\langle M, e_B\rangle$ be a basic construction for $B \subset M$. On the subset $\mathfrak{n}_{\mathrm{Tr}} \supset MeM$, we can define a non-degenerate inner product by

$$\langle x, y\rangle := \mathrm{Tr}(y^*x)$$

On the subset $Me_BM \subset \mathfrak{n}_{\mathrm{Tr}}$, which is dense with respect to the induced Hilbert norm, we have

$$\langle xe_By, ze_Ba\rangle \mathrm{Tr}((ze_Ba)^*(xe_By)) = \tau(ya^*\mathrm{E}_B(z^*x))\,,$$

for $x, y, z, a \in M$. This only depends on the properties of Tr and not on the particular realisation of $\langle M, e_B\rangle$. Denoting the Hilbert space completion of $\mathfrak{n}_{\mathrm{Tr}}$ with respect to this inner product by $\mathrm{L}^2(\langle M, e_B\rangle, \mathrm{Tr})$, we can represent $\langle M, e_B\rangle$. On the dense *-subalgebra Me_BM, this representation does not depend on concrete realisation of $\langle M, e_B\rangle$, which suffices to prove its uniqueness. Since the Tr is characterised by a relation between (M, τ) and e_B, we see that it is also unique. □

2.5 Bimodules

Definition 2.5.1. Let M, N be von Neumann algebras. An M-N-bimodule is a Hilbert space $\mathcal{H}$ with a normal *-representation $\lambda : M \to \mathcal{B}(\mathcal{H})$ and a normal *-antirepresentation $\rho : N \to \mathcal{B}(\mathcal{H})$ such that $[\lambda(x), \rho(y)] = 0$ for all $x \in M$, $y \in N$.

Example 2.5.2.

- Let M be a tracial von Neumann algebra. Then $\mathrm{L}^2(M)$ is an M-M-bimodule equipped with the actions $x\hat{y}z = \widehat{xyz}$ for all $x, y, z \in M$. We use the fact that, if τ denotes the trace of M, then

$$\|yz\|_2^2 = \tau((yz)^*yz) = \tau(z^*y^*yz) = \tau(yzz^*y^*) \le \|zz^*\|\tau(yy^*) = \|z\|^2\|y\|_2^2$$

- More generally, if $A, B \subset M$, then $\mathrm{L}^2(M)$ is an A-B-bimodule, after restricting the left and right action of M.
- If Γ is a discrete group and $\pi : \Gamma \to \mathcal{U}(H_\pi)$ is a unitary representation of Γ, then $H_\pi \otimes \ell^2(\Gamma)$ is an $\mathrm{L}(\Gamma)$-$\mathrm{L}(\Gamma)$-bimodule with left and right action induced by $\pi \otimes \lambda$ and $1 \otimes \rho$. It is necessary to invoke Fell's absorption property to see this.

Proposition 2.5.3. *Let M be a tracial von Neumann algebra and $A, B \subset M$. Then $p \mapsto p\mathrm{L}^2(M)$ establishes a one-to-one correspondence between projections in $\langle M, e_B\rangle \cap A'$ and A-B-subbimodules of $\mathrm{L}^2(M)$.*

Proof. There is a one-to-one correspondence between closed subspaces of $\mathrm{L}^2(M)$ and projections in $\mathcal{B}(\mathrm{L}^2(M))$ given by $p \mapsto p\mathrm{L}^2(M)$. If $p \in \langle M, e_B\rangle \cap A' = (JBJ' \cap \mathcal{B}(\mathrm{L}^2)) \cap A'$, then p is an A-B-linear map, and hence $p\mathrm{L}^2(M)$ is an A-B-subbimodule of $\mathrm{L}^2(M)$. Vice verse, if $p\mathrm{L}^2(M)$ is an A-B-subbimodule of $\mathrm{L}^2(M)$ for some projection $p \in \mathcal{B}(\mathrm{L}^2(M))$, then for all $\xi \in \mathrm{L}^2(M)$ and all $a \in A$, $b \in B$, we have

$$ap\xi = pap\xi$$
$$JbJp\xi = pJbJp\xi\,.$$

This shows that $ap = pap$ and $JbJp = pJbJp$ for all $a \in A$, $b \in B$. So $p \in \mathcal{B}(\mathrm{L}^2(M)) \cap A' \cap JBJ' = \langle M, e_B\rangle \cap A'$. □

The next proposition shows that we have a good notion of dimension for bimodules of tracial von Neumann algebras. This will be fixed in Definiton 2.5.5.

Proposition 2.5.4. *Let M be a tracial von Neumann algebra and ${}_McH$ a countably generated left M-module. Picking an M-module isomorphism $\mathcal{H} \cong (\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M))p$ with $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$ as in Proposition 2.3.3, the number $(\mathrm{Tr} \otimes \tau)(p)$ does only depend on ${}_McH$.*

Similarly, if $\mathcal{H}_M$ is a countably generated right M-module, then the number $(\mathrm{Tr} \otimes \tau)(p)$ for an M-module isomorphism $\mathcal{H} \cong p(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M))$ does only depend on $\mathcal{H}$.

Proof. It suffices to prove the statement for left M-modules only. Let $U : \big(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)\big)p \to \big(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)\big)q$ be a isomorphism between the two left M-modules defined by $p, q \in \mathcal{B}(\mathrm{L}^2(\mathbb{N}))\overline{\otimes}M$. Then U is a partial isometry in $\mathcal{B}(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(B)) \cap B' = \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}JBJ$ with support projection JpJ and image projection JqJ. In particular, we get

$$(\mathrm{Tr} \otimes \tau)(p) = (\mathrm{Tr} \otimes \tau)(JU^*UJ) = (\mathrm{Tr} \otimes \tau)(JUU^*J) = (\mathrm{Tr} \otimes \tau)(q)\,.$$

This finishes the proof of the proposition. □

Definition 2.5.5. Let M be a tracial von Neumann algebra and let ${}_M\mathcal{H}$ be a countably generated left M-module. The number $(\mathrm{Tr} \otimes \tau)(p)$ for an M-module isomorphism $\mathcal{H} \cong (\ell^2(\mathbb{N}))p$ with $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$ is called the (left) dimension of ${}_M\mathcal{H}$. It is denoted by $\dim_{M-}\mathcal{H}$.

If $\mathcal{H}_M$ is a countably generated right M-module and $\mathcal{H} \cong p\big((\ell^2(\mathbb{N})) \otimes \mathrm{L}^2(M)\big)$ with $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$, then $(\mathrm{Tr} \otimes \tau)(p)$ is called the (right) dimension of $\mathcal{H}_M$ and we denote it by $\dim_{-M}\mathcal{H}$.

If both M, N are tracial von Neumann algebras and $\mathcal{H}$ is an M-N-bimodule, then $(\dim_{M-}\mathcal{H} \cdot \dim_{-N}\mathcal{H})^{1/2}$ is called the index of $\mathcal{H}$.

Remark 2.5.6. A finite dimensional module does not need to be finitely generated. However, for modules over factors this is true.

Let for example f be a measurable integer valued integrable and unbounded function on $[0,1]$. Choose projections p_n of rank n in $\mathcal{B}(\ell^2(\mathbb{N}))$. Let $p \in \mathrm{L}^\infty([0,1]) \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ be defined by $p(t) = p_n$ if $f(t) = n$. Then $(\mathrm{L}^2([0,1]) \otimes \ell^2(\mathbb{N}))p$ is a finite dimensional $\mathrm{L}^\infty([0,1])$-module, but it is not finitely generated.

Proposition 2.5.7. *Let M, N be von Neumann algebras and assume that M is tracial. If ${}_M\mathcal{H}_N$ is a non-zero M-N-bimodule of finite left dimension, then $\mathcal{H}$ contains a non-zero M-N-subbimodule that is finitely generated as a left module. Similarly, every non-zero N-M-bimodule of finite right dimension contains a non-zero N-M-subbimodule that is finitely generated as a right M-module.*

Proof in the case M is a factor. It suffices to consider the case of an M-N-bimodule ${}_M\mathcal{H}_N$ of finite left dimension. We assume that M is a factor and prove that $\mathcal{H}$ is finitely generated as a left M-module. Assume that M is a factor and write $\mathcal{H} \cong (\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M))p$ for some non-zero projection $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$ that satisfies $(\mathrm{Tr} \otimes \tau)(p) < \infty$. Since M is a factor, also $\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$ is a factor. Take $n \geq (\mathrm{Tr}\otimes\tau)(p)$. The projection $p_N : \ell^2(\mathbb{N}) \to \ell^2(\{0, \dots, n-1\})$. We have $p \prec p_n \otimes 1$ or $p \succ p_n \otimes 1$ by Proposition 1.5.19. Since $(\mathrm{Tr} \otimes \tau)(p_n \otimes 1) = n \geq (\mathrm{Tr} \otimes \tau)(p)$, we conclude that $p \prec p_n \otimes 1$. So $\mathcal{H}$ is isomorphic with a M-subbimodule of $\mathbb{C}^n \otimes \mathrm{L}^2(M)$. Clearly, ${}_M(\mathbb{C}^n \otimes \mathrm{L}^2(M))$ is finitely generated as a left M-module. Further, if $\mathcal{K} \leq \mathbb{C}^n \otimes \mathrm{L}^2(M)$ is any M-submodule, then there is an M-linear projection of from $\mathbb{C}^n \otimes \mathrm{L}^2(M)$ onto $\mathcal{K}$. The image of a finite generating set of $\mathbb{C}^n \otimes \mathrm{L}^2(M)$ under this projection is a finite generating set for $\mathcal{K}$. This finishes the proof in case M is a factor. ☐

Remark 2.5.8. The proof of the general case uses the centre valued trace $\mathrm{E}_{\mathcal{Z}} : M \to \mathcal{Z}(M)$, which is by definition the trace preserving conditional expectation onto the centre. Write ${}_M\mathcal{H}_N \cong (\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M))p$ for some finite trace projection $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$ and a unital *-homomorphism $N \to p(\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M)q$. Writing $\mathcal{Z}(M) \cong \mathrm{L}^\infty(X)$ and applying $\mathrm{Tr}\otimes\mathrm{E}_{\mathcal{Z}}$ to the finite trace projection p, we obtain an integrable element over X. We can cut this element by a suitable projection in $\mathrm{L}^\infty(X)$ to obtain a non-zero bounded function. Cutting p by the same projection in $\mathrm{L}^\infty(X)$, one can prove that the resulting associated left M-module is finitely generated and it remains an M-N-bimodule in fact. However, we did not develop the necessary extension of comparison of projections from Proposition 1.5.19 in order to actually prove this.

We finish this section by linking the notion of dimension to Jones' basic construction.

Proposition 2.5.9. *Let M be a tracial von Neumann algebra and $B \subset M$ a von Neumann subalgebra. Let $p \in \langle M, e_B\rangle$ be a projection. Then the dimension of the right B-module $p\mathrm{L}^2(M)$ equals $\mathrm{Tr}(p)$.*

Proof. First note that $p\mathrm{L}^2(M)$ is indeed a right B-module by Proposition 2.5.3 (disguised as a $\mathbb{C}$-B-bimodule). Let $U : \mathrm{L}^2(M) \to q(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(B))$ be a B-module isomorphism, which satisfies as in the proof of the existence of $\langle M, e_B\rangle$ in Theorem 2.4.2 the property $\mathrm{Tr} = (\mathrm{Tr}_{\ell^2(\mathbb{N})} \otimes \tau) \circ (\mathrm{Ad}\, U)$. Then

$$\mathrm{Tr}(p) = (\mathrm{Tr}_{\ell^2(\mathbb{N})} \otimes \tau)(UpU^*) = \dim_{-B} UpU^* q(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(B)) = \dim_{-B} p\mathrm{L}^2(M)\,.$$

☐

2.6 Proof of Popa's intertwining theorem

Theorem 2.6.1. *Let $A, B \subset M$ be von Neumann subalgebras of a tracial von Neumann algebra. Then the following two statements are equivalent.*

(i) There is no sequence of unitaries $(u_n)_n$ in A such that for all $a, b \in M$ we have $\|\mathrm{E}_B(au_nb)\|_2 \to 0$

(ii) There is a non-zero element $x \in (\langle M, e_B\rangle \cap A')^+$ which has finite trace.

(iii) There is an A-B-subbimodule ${}_A\mathcal{H}_B \subset \mathrm{L}^2(M)$ which has finite right dimension.

(iv) There is a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$, a non-zero *-homomorphism $\varphi : A \to p(\mathrm{M}_n(\mathbb{C}) \otimes B)p$ and a non-zero partial isometry $v \in (\mathrm{M}_{1,n}(\mathbb{C}) \otimes M)p$ such that $v\varphi(a) = av$ for all $a \in A$.

If either of the conditions of this theorem is satisfies, we say that a corner of A embeds into a corner of B inside M and we write $A \prec_M B$.

Proof. We prove (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i).

Assume that (i) holds. Then there is some $\varepsilon > 0$ and some finite subset $F \subset M$ such that for all $u \in \mathcal{U}(A)$ we have

$$\| \sum_{x,y \in F} \mathrm{E}_B(xuy^*) \|_2 \geq \varepsilon \, .$$

Consider in $\mathrm{L}^2(\langle M, e_B \rangle, \mathrm{Tr})$ the set

$$C = \overline{\mathrm{conv}}^{\| \ \|_{2,\mathrm{Tr}}} \{ \sum_{x,y \in F} ux^* e_B y u^* \mid u \in \mathcal{U}(A) \} \, .$$

Since $\| \ \|_{2,\mathrm{Tr}}$ is finite and bounded on $\mathrm{conv}\{\sum_{x,y\in F} ux^* e_B y u^* \mid u \in \mathcal{U}(A)\}$, we see that C is a $\| \ \|_{2,\mathrm{Tr}}$-closed subset of $\mathrm{L}^2(\langle M, e_B \rangle, \mathrm{Tr})$. So Proposition 2.1.6 says that there is a unique element $m \in C$ of minimal $\| \ \|_{2,\mathrm{Tr}}$. In particular, $\mathrm{Tr}(m^* m) < \infty$. We have

$$\begin{aligned} \mathrm{Tr}((\sum_{x,y \in F} ux^* e_B y u^*)(\sum_{z,a \in F} z^* e_B a)) &= \mathrm{Tr}(\sum_{x,y,z,a \in F} e_B a u x^* e_B \mathrm{E}_B(yu^* z^*)) \\ &= \mathrm{Tr}(\mathrm{E}_B(\sum_{x,y\in F} xuy^*) e_B \mathrm{E}_B(\sum_{x,y\in F} xu^* y^*)) \\ &= \| \mathrm{E}_B(\sum_{x,y \in F} xuy^*) \|_2^2 \\ &\geq \varepsilon \, , \end{aligned}$$

which shows that also

$$\mathrm{Tr}(m(\sum_{x,y\in F} x^* e_B y)) \geq \varepsilon \, .$$

So $m \neq 0$. Further, $uCu^* = C$ for all $u \in \mathcal{U}(A)$ implies that $m \in \langle M, e_B \rangle \cap A'$. So $m^* m \in \big(\langle M, e_B \rangle \cap A'\big)^+$ is a non-zero element of finite trace. This proves (ii).

Now assume that (ii) holds. Let $x \in \big(\langle M, e_B \rangle \cap A'\big)^+$ be a non-zero element with finite trace. Take $\varepsilon > 0$ such that $0 \neq p = \mathbb{1}_{[\varepsilon, \infty)}(x)$. Then $p\big(\langle M, e_B\rangle \cap A'\big)^+$ and $p \leq \frac{1}{\varepsilon} x$ implying that it has finite trace. According to Proposition 2.5.9, the A-B-bimodule $p\mathrm{L}^2(M)$ has right dimension $\dim_{-B}(p\mathrm{L}^2(M)) = \mathrm{Tr}(p) < \infty$. This proves (iii).

Let us now assume (iii) and let $\mathcal{K} \leq {}_A\mathrm{L}^2(M)_B$ be an A-B-bimodule of finite right dimension. By Proposition 2.5.7, we may assume that $\mathcal{K}$ is finitely generated as a right B-module. We obtain a normal unital *-homomorphism $\varphi : A \to p(\mathrm{M}_n(\mathbb{C}) \otimes B)p$ and an isomorphism $V : {}_{\varphi(A)} p\big(\mathbb{C}^n \otimes \mathrm{L}^2(B)\big)_B \to \mathcal{K}$ of A-B-bimodules. Consider the vectors $\xi_i := V(p(\delta_i \otimes \hat{1})) \in \mathrm{L}^2(M)$. They satisfy for $a \in A$

$$\begin{aligned} a\xi_i &= aV(p(\delta_i \otimes \hat{1})) \\ &= V\varphi(a)(\delta_i \otimes \hat{1}) \\ &= \sum_j V p(\delta_j \otimes \widehat{\varphi(a)}_{ji}) \\ &= \sum_j \xi_j \varphi(a)_{ji} \, . \end{aligned}$$

If we set $\xi = \sum_i e_{1i} \otimes \xi_i \in \mathrm{M}_{1,n} \otimes \mathrm{L}^2(M)$, then this implies

$$\xi\varphi(a) = \big(e_{1i} \otimes \xi_i\big)\varphi(a) = \sum_{i,j} e_{1j} \otimes \xi_i \varphi(a)_{ij} = \sum_j e_{1j} \otimes a\xi_j = a\xi\,,$$

for all $a \in A$.

Using a generalisation of the polar decomposition from Proposition 1.5.16, we obtain a decomposition $\xi = v|\xi|$ for the vector $\xi \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes \mathrm{L}^2(M) \subset \mathrm{L}^2(\mathrm{M}_n(\mathbb{C}) \otimes M)$, where $|\xi| \in \overline{M^+}^{\|\ \|_2}$ and $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes M$ is a partial isometry. Using a uniqueness argument for the polar decomposition, we obtain $av = v\varphi(a)$ for all $a \in A$. In particular, $v = vp \in \big(\mathrm{M}_{1,n}(\mathbb{C}) \otimes M\big)p$. This proves (iv).

Assume (iv) and $\varphi : A \to p\big(\mathrm{M}_n(\mathbb{C}) \otimes B\big)p$ be a non-zero *-homomorphism and $v \in \mathrm{M}_{1,n}(M)p$ be a non-zero partial isometry satisfying $v\varphi(a) = av$ for all $a \in A$. If $(u_n)_n$ is a sequence of unitaries in A, then considering matrices over M, we have

$$\mathrm{E}_B(v^* u_n v) = \mathrm{E}_B(v^* v\varphi(u_n)) = \mathrm{E}_B(v^* v)\varphi(u_n)\,,$$

showing that matrix elements of $\mathrm{E}_B(v^* u_n v)$ do not go to 0 in $\|\ \|_2$. This shows (i). ☐

Lemma 2.6.2. *Let $A \subset M$ be a Cartan subalgebra in a finite factor. If $p, q \in A$ are non-zero projections, then there is a unitary $u \in \mathcal{N}_M(A)$ such that $upu^* q \neq 0$.*

Proof. Denote by $z = \bigvee_{u \in \mathcal{N}_M(A)} upu^*$ the smallest projection in A containing all $\mathcal{N}_M(A)$ conjugates of p. Then z is $\mathcal{N}_M(A)$-invariant and hence central in the factor M. It follows that $z = 1$. In particular $zq \neq 0$, implying that there is $u \in \mathcal{N}_M(A)$ such that $upu^* q \neq 0$. ☐

Theorem 2.6.3. *If M is a finite factor and $A, B \subset M$ are Cartan subalgebras, then $A \prec_M B$ implies that there is a unitary $u \in M$ such that $uAu^* = B$.*

Proof. It remains to show that if $A, B \subset M$ are Cartan subalgebras of a finite factor, then $A \prec B$ implies that there is a unitary $u \in M$ satisfying $uAu^* = B$.

We first prove that there is a non-zero partial isometry $v \in M$ such that (a) $v^* v \in A$, (b) $vv^* \in B$, and (c) $v^* A v \subset B$. Let $p \in \mathrm{M}_n(\mathbb{C}) \otimes B$ be a projection, $\varphi : A \to p(\mathrm{M}_n(\mathbb{C}) \otimes B)p$ a *-homomorphism and $v \in \big(\mathrm{M}_{1,n}(\mathbb{C}) \otimes M\big)p$ a partial isometry satisfying $v\varphi(a) = av$ for all $a \in A$. We may assume that φ is unital. Since $\varphi(A) \subset \mathrm{M}_n(\mathbb{C}) \otimes B$ is an abelian subalgebra, it can be conjugated into $\mathbb{C}^n \otimes B$. In particular, $p \in \mathbb{C}^n \otimes B$ and we can cut down by a rank one projection in $\mathbb{C}^n$ in order to assume that $n = 1$.

Put $e := vv^*$. Now $avv^* = v\varphi(a)v^* = vv^* a$, shows that $e \in A' \cap M = A$. Let $f := v^* v$. Then $f \in \varphi(A)' \cap pMp$. Further $f(\varphi(A)' \cap pMp)f = v^*(A' \cap eMe)v = v^* A v$ is abelian, which uses the fact that $e \in A$. We use a slight extension of Theorem 1.5.21: the projection f is abelian in $\varphi(A)' \cap pMp$, generalising the notion of minimal projections to not necessarily factorial von Neumann algebras. An extension of considerations made in the proof of Theorem 1.5.21 shows that there is a projection $f' \in pB$ such that $f \sim f'$, that is, there is a partial isometry $w \in \varphi(A)' \cap pMp$ such that $ww = f$ and $w^* w = f'$. Now vw is a non-zero partial isometry in M such that $\operatorname{supp} vw = w^* v^* v w = w^* f w = f' \in Bp$ and $\operatorname{im} vw = \operatorname{im} v \in A$. Further $(vw)\varphi(a) = v\varphi(a)w = a(vw)$ for all $a \in A$. This implies $(vw)^* A(vw) = f'\varphi(A) \subset B$.

Let (v_i) be maximal family of partial isometries in M such that $v_i^* v_i \in A$ are pairwise orthogonal, $v_i v_i^* \in B$ are pairwise orthogonal and $v_i A v_i^* \subset B$ for all i. If $\sum_i v_i^* v_i \neq 1$, then also $\sum_i v_i v_i^* \neq 1$, since M

is finite. Write $r = 1 - \sum_i v_i^* v_i$ and $s = 1 - \sum_i v_i v_i^*$. By Lemma 2.6.2, there are unitaries $u \in \mathcal{N}_M(A)$ and $w \in \mathcal{N}_M(B)$ such that $uru^* \operatorname{supp} v \neq 0$ and $w^* sw(vuru^* v^*) \neq 0$. Then $swvur$ is a non-zero partial isometry whose support projection lies in A and is contained in r and whose image projection lies in B and is contained in s. This contradicts maximality of the family $(v_i)_i$. So $u := \sum_i v_i$ is a unitary in M satisfying $uAu^* \subset B$. Since uAu^* is maximal abelian in M and B is abelian, we have $uAu^* = B$ in fact. This finishes the proof of the theorem. □

3 A rigidity result

In this section we are going to give examples of actions $\Gamma \curvearrowright (X, \mu)$ which to a certain extend can up to orbit equivalence be recovered from their group-measure space von Neumann algebras. This will give rise to examples of non-isomorphic group-measure space constructions.

3.1 Completely positive maps

Definition 3.1.1. Let M, N be von Neumann algebras. A linear map $\varphi : M \to N$ is called completely positive (abbreviated "cp map"), if all amplifications $\varphi \otimes \mathrm{id} : M \otimes \mathrm{M}_n(\mathbb{C}) \to N \otimes \mathrm{M}_n(\mathbb{C})$ are positive maps. If $\varphi(1) = 1$, then φ is called a unital completely positive map (abbreviated "ucp map").

Example 3.1.2.

- Every *-homomorphism between von Neumann algebras is completely positive.
- Every positive functional on a von Neumann algebra is a completely positive map.
- If $M \subset \mathcal{B}(H)$ and $T \in \mathcal{B}(H)$, then $M \ni x \mapsto TxT^*$ is a completely positive map.

The next two propositions show that the previous examples suffice to describe all normal trace preserving ucp maps between von Neumann algebras.

Proposition 3.1.3. *Let M, N be tracial von Neumann algebras and let $\varphi : M \to N$ be a normal trace preserving ucp map. Then*

$$\langle x \otimes \xi, y \otimes \eta \rangle := \langle \varphi(y^* x)\xi, \eta \rangle$$

defines an inner product on $M \otimes_{alg} \mathrm{L}^2(N)$. The completion of $M \otimes_{alg} \mathrm{L}^2(N)$ with respect to this inner product is an M-N-bimodule with generating vector $\xi_\varphi = 1 \otimes \hat{1}$. This vector has the property that

$$\langle x\xi_\varphi y, \xi_\varphi \rangle = \tau(\varphi(x)y)$$

for all $x \in M$ and $y \in N$. In particular,

$$\langle x\xi_\varphi, \xi_\varphi \rangle = \tau(x)\,, \qquad \langle \xi_\varphi y, \xi_\varphi \rangle = \tau(y)\,.$$

Proof. We first show that $\langle x \otimes \xi, y \otimes \eta \rangle = \langle \varphi(y^* x)\xi, \eta \rangle$ defines an inner product on $M \otimes_{\mathrm{alg}} \mathrm{L}^2(N)$. It is clear that it is sesquilinear, so we only have to show positive definiteness. Let $\sum_{i=1}^n x \otimes \xi_i \in M \otimes_{\mathrm{alg}} \mathrm{L}^2(N)$ be a vector, written in such a way that $\xi_1, \dots, \xi_n$ are pairwise orthogonal. Consider the amplification $\varphi_n : M \otimes \mathrm{M}_n(\mathbb{C}) \to M \otimes \mathrm{M}_n(\mathbb{C})$, which is positive by assumption. So the element

$$\varphi_n((x_i^* x_j)_{i,j}) = \varphi_n(\begin{pmatrix} 0 & \cdots & x_1^* \\ & \vdots & \\ 0 & \cdots & x_n^* \end{pmatrix}\begin{pmatrix} 0 & \cdots & 0 \\ & \cdots & \\ x_1 & \cdots & x_n \end{pmatrix}) = \varphi_n(\begin{pmatrix} 0 & \cdots & 0 \\ & \cdots & \\ x_1 & \cdots & x_n \end{pmatrix}^* \begin{pmatrix} 0 & \cdots & 0 \\ & \cdots & \\ x_1 & \cdots & x_n \end{pmatrix})$$

is positive. We obtain that

$$\langle \sum_i x_i \otimes \xi_i, \sum_i x_i \otimes \xi_i \rangle = \sum_{i,j} \langle \varphi(x_i^* x_j)\xi_j, \xi_i \rangle = \langle \varphi_n((x_i^* x_j)_{i,j}) \sum_i \xi_i \otimes \delta_i, \sum_i \xi_i \otimes \delta_i \rangle \geq 0\,,$$

where $\delta_1, \dots, \delta_n$ denotes the standard basis of $\mathbb{C}^n$. This proves positive definiteness. Denote the completion of $M \otimes_{\mathrm{alg}} \mathrm{L}^2(N)$ with respect to this inner product by $\mathcal{H}_\varphi$.

For $m \in M$, $n \in N$ we have

$$\begin{aligned}\|mx \otimes \xi n\|^2 &= \langle \varphi(x^* m^* m x)\xi n, \xi n\rangle \\ &\leq \|m^* m\| \langle \varphi(x^* x)\xi n, \xi n\rangle \\ &= \|m\|^2 \langle \varphi(x^* x)^{1/2}\xi n, \varphi(x^* x)^{1/2}\xi n\rangle \\ &\leq \|m\|^2 \|n\|^2 \langle \varphi(x^* x)^{1/2}\xi, \varphi(x^* x)^{1/2}\xi\rangle \\ &= \|m\|^2 \|n\|^2 \|x \otimes \xi\|^2 .\end{aligned}$$

So the natural representations of M and N on $\mathcal{H}$ are bounded. Further, putting $\xi_\varphi := 1 \otimes \hat{1} \in \mathcal{H}_\varphi$, we have

$$\langle x\xi_\varphi y, \xi_\varphi\rangle = \langle \varphi(x)\hat{y}, \hat{1}\rangle = \tau(\varphi(x)y) .$$

In particular,

$$\langle x\xi_\varphi, \xi_\varphi\rangle = \tau(\varphi(x)) = \tau(x) , \qquad \langle \xi_\varphi y, \xi_\varphi\rangle = \tau(\varphi(1)y) = \tau(y) ,$$

since φ is trace preserving and unital. This also proves that the representations of M and N on $\mathcal{H}_\varphi$ are normal. □

Proposition 3.1.4. *Let M, N be tracial von Neumann algebras and $\mathcal{H}$ an M-N-bimodule with a vector $\xi \in \mathcal{H}$ such that for all $x \in M$, $y \in N$ we have*

$$\langle x\xi, \xi\rangle = \tau(x) , \qquad \langle \xi y, \xi\rangle = \tau(y) .$$

Then ξ is a bounded vector for N and considering $M \subset \mathcal{B}(\mathcal{H})$, the map $x \mapsto \mathrm{L}_\xi^* x \mathrm{L}_\xi$ defines a normal trace preserving ucp map $\varphi_\xi : M \to N$.

Proof. For $y \in N$ we have

$$\|\xi y\|^2 = \langle \xi y y^*, \xi\rangle = \tau(yy^*) = \|y\|_2^2 ,$$

showing that ξ is bounded for N and L_ξ is isometric. For $x \in M$, $y, n_1, n_2 \in N$ we have

$$\begin{aligned}\langle JyJ\mathrm{L}_\xi^* x \mathrm{L}_\xi \hat{n}_1, \hat{n}_2\rangle &= \langle \mathrm{L}_\xi^* x \mathrm{L}_\xi \hat{n}_1, \widehat{n_2 y^*}\rangle \\ &= \langle x\mathrm{L}_\xi \hat{n}_1, \mathrm{L}_\xi \widehat{n_2 y^*}\rangle \\ &= \langle x\xi n_1, \xi n_2 y^*\rangle \\ &= \langle x\xi n_1 y, \xi n_2\rangle \\ &= \langle \mathrm{L}_\xi^* x \mathrm{L}_\xi JyJ\hat{n}_1, \hat{n}_2\rangle .\end{aligned}$$

This shows that $(\mathrm{Ad}\,\mathrm{L}_\xi^*)(M) \subset JNJ' \cap \mathcal{B}(\mathrm{L}^2(N)) = N$. So considering $M \subset \mathcal{B}(\mathcal{H})$ we can define $\varphi_\xi = (\mathrm{Ad}\,\mathrm{L}_\xi^*) : M \to N$. It is clear that φ_ξ is a normal completely positive map. Further, $\varphi_\xi(1) = \mathrm{L}_\xi^* \mathrm{L}_\xi = 1$, since L_ξ is isometric. So φ_ξ is unital. Finally, for all $x \in M$, we have

$$\tau(\mathrm{L}_\xi^* x \mathrm{L}_\xi) = \langle \mathrm{L}_\xi^* x \mathrm{L}_\xi \hat{1}, \hat{1}\rangle = \langle x\xi, \xi\rangle = \tau(x) ,$$

showing that φ_ξ is trace preserving. This finishes the proof of the proposition. □

Definition 3.1.5. Let M, N be tracial von Neumann algebras. If $\varphi : M \to N$ is a normal trace preserving ucp map, then we denote by $\xi_\varphi \in \mathcal{H}_\varphi$ the M-N-bimodule with distinguished vector associated with φ by Proposition 3.1.3.

If $\mathcal{H}$ is an M-N-bimodule with a generating tracial vector $\xi \in \mathcal{H}$, then φ_ξ denotes the normal trace preserving ucp map $\varphi_\xi : M \to N$ associated with $\xi \in \mathcal{H}$ by Proposition 3.1.4.

Summarising Propositions 3.1.3 and 3.1.4, we obtain the following correspondence between ucp maps and bimodules.

Theorem 3.1.6. *Let M, N be tracial von Neumann algebras. The assignment $\varphi \mapsto (\mathcal{H}_\varphi, \xi_\varphi)$ with inverse $(\mathcal{H}, \xi) \mapsto \varphi_\xi$ is a one-to-one correspondence between*

(i) normal trace preserving ucp maps $\varphi : M \to N$, and

(ii) M-N-bimodules $\mathcal{H}$ with a fixed generating tracial vector $\xi \in \mathcal{H}$.

Proof. We have to show that (i) $\varphi_{\xi_\varphi} = \varphi$ and (ii) $(\mathcal{H}_{\varphi_\xi}, \xi_{\varphi_\xi}) = (\mathcal{H}, \xi)$. Let $x \in M$ and $y \in N$. Then

$$\begin{aligned}\tau(\varphi_{\xi_\varphi}(x)y) &= \tau(\mathrm{L}_{\xi_\varphi}^* x \mathrm{L}_{\xi_\varphi} y) \\ &= \langle \mathrm{L}_{\xi_\varphi}^* x \mathrm{L}_{\xi_\varphi} y \hat{1}, \hat{1} \rangle \\ &= \langle x \xi_\varphi y, \xi_\varphi \rangle \\ &= \tau(\varphi(x)y) ,\end{aligned}$$

using Proposition 3.1.3 for the last equality. This shows (i).

Calculating in $\mathcal{H}_{\varphi_\xi}$, we obtain for $x \in M$ and $y \in N$ that

$$\begin{aligned}\|x \otimes \hat{y}\|^2 &= \langle x \otimes \hat{y}, x \otimes \hat{y} \rangle \\ &= \langle \varphi_\xi(x^* x)\hat{y}, \hat{y} \rangle \\ &= \langle \mathrm{L}_\xi^* x^* x \mathrm{L}_\xi \hat{y}, \hat{y} \rangle \\ &= \langle x\xi y, x\xi y \rangle \\ &= \|x \xi y\|^2 .\end{aligned}$$

So $M \otimes \mathrm{L}^2(N) \ni x \otimes \hat{y} \mapsto x\xi y \in \mathcal{H}$ extends to a unitary between $\mathcal{H}_{\varphi_\xi}$ and $\mathcal{H}$. It is clear that this unitary intertwines the M and the N action and that it maps $1 \otimes \hat{1}$ to ξ. This proves (ii). ☐

3.2 Positive type functions

Definition 3.2.1. Let X be a set and $\varphi : X \times X \to \mathbb{C}$ a kernel on X. We say that φ is of positive type, if for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and all $x_1, \dots, x_n \in X$ we have

$$\sum_{i,j=1}^n \lambda_i \overline{\lambda_j} \varphi(x_i, x_j) \geq 0$$

Let Γ be a discrete group. A function $\varphi : \Gamma \to \mathbb{C}$ is called of positive type, if $(g,h) \mapsto \varphi(h^{-1}g)$ is a kernel of positive type on Γ. More explicitly, φ is a function of positive type if for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and all $g_1, \dots, g_n \in \Gamma$ we have

$$\sum_{i,j=1}^n \lambda_i \overline{\lambda_j} \varphi(g_j^{-1} g_i) \geq 0 .$$

We call φ normalised if $\varphi(e) = 1$.

The next theorem gives an analogue of the GNS-construction for kernels and functions of positive type.

Theorem 3.2.2. *Let $\varphi : X \times X \to \mathbb{C}$ be a kernel of positive type on a set X. Then there is a Hilbert space H and a map $\xi : X \to H : x \mapsto \xi_x$ whose image generates H such that*

$$\varphi(x, y) = \langle \xi_x, \xi_y \rangle$$

for all $x, y \in X$.

Conversely, if $\xi : X \to H$ is a map into a Hilbert space whose image generates H, then $(x, y) \mapsto \langle \xi_x, \xi_y \rangle$ is a kernel of positive type on X. So there is a one-to-one correspondence between

- *Kernels of positive type on X, and*
- *maps $X \to H$ whose image generates H (up to unitary conjugacy).*

Let Γ be a discrete group and φ a function of positive type on Γ. Then there is a cyclic unitary representation (H, ξ) of Γ such that

$$\varphi(g) = \langle g\xi, \xi \rangle$$

for all $g \in \Gamma$.

Conversely, if (H, ξ) is a cyclic unitary representation of Γ, then $g \mapsto \langle g\xi, \xi \rangle$ is a function of positive type on Γ. So there is a one-to-one correspondence between

- *Positive type functions on Γ, and*
- *cyclic unitary representations of Γ (up to unitary conjugacy).*

Proof. Let X be some set and $\varphi : X \times X \to \mathbb{C}$ be a kernel of positive type. On $\mathbb{C}X$ define an sesquilinear form by

$$\langle \delta_x, \delta_y \rangle := \varphi(x, y) \, .$$

This defines an inner product, since we have

$$\langle \sum_{i=1}^n \lambda_i \delta_{x_i}, \sum_{i=1}^n \lambda_i \delta_{x_i} \rangle = \sum_{i,j=1}^n \lambda_i \overline{\lambda_j} \varphi(x_i, x_j) \geq 0 \, ,$$

for all elements $\sum_{i=1}^n \lambda_i \delta_{x_i} \in \mathbb{C}X$. After separation-completion we obtain a Hilbert space H and a map $\xi : X \to H : x \mapsto \xi_x := \hat{\delta}_x$ satisfying

$$\varphi(x, y) = \langle \xi_x, \xi_y \rangle$$

for all $x, y \in X$. Note that H is generated by the image of ξ.

If $\xi : X \to H$ is some map into a Hilbert space, then reversing the previous calculation, we see that $(x, y) \mapsto \langle \delta_x, \delta_y \rangle$ is of positive type.

Now assume that Γ is a group and φ is a function of positive type on Γ. We obtain a Hilbert space H and a map $\xi : \Gamma \to H$ whose image generates H and such that

$$\varphi(h^{-1}g) = \langle \xi_g, \xi_h \rangle \, ,$$

for all $g, h \in \Gamma$. For $g \in \Gamma$, the rule $g\xi_h := \xi_{gh}$ defines a unitary representation of Γ on H, since

$$\langle \xi_{gh}, \xi_{gk} \rangle = \varphi((gk)^{-1}gh) = \varphi(k^{-1}h) = \langle \xi_h, \xi_k \rangle \, ,$$

for all $h, k \in \Gamma$. Let $\xi := \xi_e$. Then ξ is a cyclic vector for the representation of Γ on H. Further, for all $g \in \Gamma$, we have

$$\varphi(g) = \langle \xi_g, \xi_e \rangle = \langle g\xi, \xi \rangle .$$

If (H, ξ) is a cyclic unitary representation of Γ, then we can apply the first part of the theorem to the map $\Gamma \to H : g \mapsto g\xi$. We obtain that $(g, h) \mapsto \langle g\xi, h\xi \rangle$ is a kernel of positive type, which implies that $g \mapsto \langle g\xi, \xi \rangle$ is a function of positive type on Γ. □

The following proposition proves some useful permanence properties for positive type functions.

Proposition 3.2.3. *Let φ, ψ be either kernels of positive type on a set X or functions of positive type on a group Γ. Then the following kernels/functions are again of positive type:*

- *$t\varphi$ for all $t \geq 0$,*
- *$\varphi + \psi$, and*
- *$\varphi \cdot \psi$.*

Proof. We only need to consider kernels of positive type. Further, the only non-obvious claim is that products of two kernels of positive type is a kernel of positive type. By Theorem 3.2.2, there are maps into Hilbert spaces $\xi : X \to H$ and $\eta : X \to K$ satisfying $\varphi(x, y) = \langle \xi_x, \xi_y \rangle$ and $\psi(x, y) = \langle \eta_x, \eta_y \rangle$. Considering the tensor product $H \otimes K$, we see that

$$(x, y) \mapsto \langle \xi_x \otimes \eta_x, \xi_y \otimes \eta_y \rangle = \varphi(x, y)\psi(x, y)$$

is a kernel of positive type on X. □

Proposition 3.2.4. *Let $\Gamma \curvearrowright X$ a pmp action and let $\varphi : \Gamma \to \mathbb{C}$ be a normalised positive type function. There is a well-defined normal $\mathrm{L}^\infty(X)$-bimodular ucp map $\Phi : \mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(X) \rtimes \Gamma$ satisfying $\Phi(u_g) = \varphi(g)u_g$ for all $g \in \Gamma$.*

Proof. We first reduce to the case where X is a point. Assume that there is a trace preserving ucp map $\Phi : \mathrm{L}(\Gamma) \to \mathrm{L}(\Gamma)$ such that $\Phi(u_g) = \varphi(g)u_g$ for all $g \in \Gamma$. Let $\Gamma \curvearrowright X$ be a pmp action. Inside $(\mathrm{L}^\infty(X) \rtimes \Gamma)\overline{\otimes}\mathrm{L}(\Gamma)$ consider $\mathrm{L}^\infty(X) \otimes 1$ and the elements $u_g \otimes u_g$. They satisfy the abstract characterisation of the group-measure space construction of Definition 1.4.17. Hence by Theorem 1.4.21, there is a unique *-homomorphism $\Delta : \mathrm{L}^\infty(X) \rtimes \Gamma \to (\mathrm{L}^\infty(X) \rtimes \Gamma)\overline{\otimes}\mathrm{L}(\Gamma)$ satisfying $\Delta(f) = f \otimes 1$ and $\Delta(u_g) = u_g \otimes u_g$ for all $f \in \mathrm{L}^\infty(X)$ and all $g \in \Gamma$. The map $\mathrm{id} \otimes \Phi$ is $\Delta(\mathrm{L}^\infty(X))$-bimodular and it satisfies

$$(\mathrm{id} \otimes \Phi)(\Delta(u_g)) = (\mathrm{id} \otimes \Phi)(u_g \otimes u_g) = \varphi(g)(u_g \otimes u_g) = \varphi(g)\Delta(u_g) .$$

In particular, $\mathrm{id} \otimes \Phi$ preserves $\Delta(\mathrm{L}^\infty(X) \rtimes \Gamma)$. So we can restrict $\mathrm{id} \otimes \Phi$ to a trace preserving ucp map $\tilde{\Phi} : \mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(X) \rtimes \Gamma$. Now $\tilde{\Phi}$ is $\mathrm{L}^\infty(X)$-bimodular and it satisfies $\tilde{\Phi}(u_g) = \varphi(g)u_g$. This finishes the proof under the assumption that Φ exists.

Theorem 3.2.2 gives a cyclic representation π of Γ on a Hilbert space H with cyclic vector ξ satisfying $\langle \pi(g)\xi, \xi \rangle = \varphi(g)$. By Fell's absorption property from Theorem 1.4.15, the representation $\pi \otimes \lambda$ of Γ on $H \otimes \ell^2(\Gamma)$ extends to a representation of $\mathrm{L}(\Gamma)$. The right regular representation $\mathrm{id} \otimes \rho$, then defines the structure of an $\mathrm{L}(\Gamma)$-$\mathrm{L}(\Gamma)$-bimodule on $H \otimes \ell^2(\Gamma)$. Note that the vector $\xi \otimes \delta_e$ is tracial and generating. So Proposition 3.1.4 says that there is a normal tracial ucp map $\Phi : \mathrm{L}(\Gamma) \to \mathrm{L}(\Gamma)$

defined by $u_g \mapsto \mathrm{Ad}(\mathrm{L}^*_{\xi\otimes\delta_e})(\pi(g)\otimes u_g)$. In order to determine $\Phi(u_g)$, it suffices to calculate $\Phi(u_g)\delta_e$. For all $h \in \Gamma$, we have

$$\begin{aligned}\langle \Phi(u_g)\delta_e, \delta_h\rangle &= \langle \mathrm{L}^*_{\xi\otimes\delta_e}(\pi(g)\otimes u_g)\mathrm{L}_{\xi\otimes\delta_e}\delta_e, \delta_h\rangle \\ &= \langle (\pi(g)\otimes u_g)\mathrm{L}_{\xi\otimes\delta_e}\delta_e, \mathrm{L}_{\xi\otimes\delta_e}\delta_h\rangle \\ &= \langle (\pi(g)\otimes u_g)(\xi\otimes\delta_e), \xi\otimes\delta_h\rangle \\ &= \langle \pi(g)\xi,\xi\rangle\langle\delta_g,\delta_h\rangle \\ &= \delta_{g,h}\varphi(g)\,.\end{aligned}$$

This implies that $\Phi(u_g) = \varphi(g)u_g$, finishing the proof of the proposition. □

3.3 The Haagerup property

The aim of this section is to introduce the Haagerup property and show that free groups of finite rank do have this property. We will finish, by showing how the Haagerup property of a group Γ is reflected in a group measure space construction $\mathrm{L}^\infty(X) \rtimes \Gamma$.

Definition 3.3.1. A group Γ is said to have the Haagerup property, if there is a sequence of positive type functions $\varphi_i : \Gamma \to \mathbb{C}$ such that $\varphi_i \in \mathrm{C}_0(\Gamma)$ for every i and $\varphi_i \to 1$ pointwise.

Recall that the free group $\mathbb{F}_n$ of rank n is the set of (possibly empty) reduced words in letters $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$ with the product defined by concatenation and successive reduction (i.e. deleting possible appearances of $x_i x_i^{-1}$ and $x_i^{-1}x_i$). Consider the graph T whose set of vertices is $\mathbb{F}_n$ as a set of vertices of a graph, and with edges (g, gx_i) for $g \in \mathbb{F}_n$. Then T is a $2n$-regular tree, on which $\mathbb{F}_n$ acts by left multiplication. T is called the Cayley graph of $\mathbb{F}_n$.

Theorem 3.3.2. *For every $n \in \mathbb{N}$, the free group $\mathbb{F}_n$ has the Haagerup property.*

Proof. Consider the action $\mathbb{F}_n \curvearrowright T$ on its Cayley graph. Denote by ρ the root of T and by $\mathrm{E}(T)$ the set of edges of T. Then

$$b(g)(e) = \begin{cases} 1 & \text{if } e \in [\rho, g\rho] \\ -1 & \text{if } \overline{e} \in [\rho, g\rho] \\ 0 & \text{otherwise} \end{cases}$$

defines a function $b : \mathbb{F}_n \to \ell^2(\mathrm{E}(T))$. We have

$$\|b(g) - b(h)\|^2 = \sum_{e\in\mathrm{E}(T)} |b(g)(e) - b(h)(e)|^2 = 2|[g\rho, h\rho]| = 2\mathrm{d}(g,h)\,,$$

where $\mathrm{d}(g,h)$ denotes the distance between $g, h \in \mathbb{F}_n$. We check that

$$b(gh) = gb(h) + b(g)\,,$$

for all $g, h \in \mathbb{F}_n$. The path first running through $[\rho, g\rho]$ and then $[g\rho, gh\rho]$ crosses all edged from $[\rho, gh\rho]$ and additionally crosses all edges from $[\rho, g\rho] \cap [g\rho, gh\rho]$ in both directions. We obtain that

$$b(gh) = \sum_{e\in[\rho,gh\rho]} \delta_e - \delta_{\overline{e}} = \sum_{e\in[rho,g\rho]} \delta_e - \delta_{\overline{e}} + \sum_{e\in[g\rho,gh\rho]} \delta_e - \delta_{\overline{e}} = b(g) + gb(h)\,.$$

So $b : \mathbb{F}_n \to \ell^2(\mathrm{E}(T))$ is a 1-cocyle. In particular, it satisfies $b(g^{-1}) = -g^{-1}b(g)$ for all $g \in \mathbb{F}_n$. This implies that

$$\|b(g^{-1}h)\| = \|b(g^{-1}) + g^{-1}b(h)\| = \|g^{-1}b(h) - g^{-1}b(g)\| = \|b(h) - b(g)\|$$

for all $g, h \in \mathbb{F}_2$. We use next the formula $\|xi - \eta\|^2 = \|\xi\|^2 + \|\eta\|^2 - 2\mathfrak{Re}\langle \xi, \eta\rangle$, which for element ξ, η in every Hilbert space. For all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and all $g_1, \ldots, g_n \in \mathbb{F}_n$, we have

$$\begin{aligned}\sum_{i,j=1}^{n} \lambda_i\overline{\lambda_j}\big(\|b(g_i)\|^2 + \|b(g_j)\|^2 - \|b(g_i) - b(g_j)\|^2\big) &= \sum_{i,j=1}^{n} 2\,\mathfrak{Re}\langle \lambda_i b(g_i), \lambda_j b(g_j)\rangle \\ &= 2\|\sum_{i=1}^{n}\lambda_i b(g_i)\|^2 \\ &\geq 0\,.\end{aligned}$$

So $(g,h) \mapsto \|b(g)\|^2 + \|b(h)\|^2 - \|b(g) - b(h)\|^2$ is a kernel of positive type on $\mathbb{F}_n$. Since products of kernels of positive type are again kernels of positive type by Proposition 3.2.3, this implies that

$$(g,h) \mapsto \mathrm{e}^{\|b(g)\|^2 + \|b(h)\|^2 - \|b(g)-b(h)\|^2} = \sum_{k\in\mathbb{N}} \frac{(\|b(g)\|^2 + \|b(h)\|^2 - \|b(g) - b(h)\|^2)^k}{k!}$$

is a kernel of positive type on $\mathbb{F}_n$. Again take $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $g_1, \ldots, g_n \in \mathbb{F}_n$. Then

$$\sum_{i,j=1}^{n} \lambda_i\overline{\lambda_j}\mathrm{e}^{-(\|b(g_i)\|^2 + \|b(g_j)\|^2)} = |\sum_{i=1}^{n}\lambda_i \mathrm{e}^{-\|b(g_i)\|^2}|^2 \geq 0\,,$$

So

$$(g,h) \mapsto \mathrm{e}^{-(\|b(g)\|^2 + \|b(h)\|^2)}$$

is a kernel of positive type on Γ. We conclude by Proposition 3.2.3 that

$$(g,h) \mapsto \mathrm{e}^{-\|b(h^{-1}g)\|^2} = \mathrm{e}^{-\|b(h)-b(g)\|^2} = \mathrm{e}^{\|b(g)\|^2 + \|b(h)\|^2 - \|b(h)-b(g)\|^2}\mathrm{e}^{-(\|b(g)\|^2 + \|b(h)\|^2)}$$

is a kernel of positive type. This means that $g \mapsto \mathrm{e}^{-\|b(g)\|^2}$ is a function of positive type on $\mathbb{F}_n$. Replacing b by $t^{1/2}b$ for some $t \geq 0$ in this argument, we obtain that that all the functions $\varphi_t(g) := \mathrm{e}^{-t\|b(g)\|^2}$, $g \in \mathbb{F}_n$ are of positive type. We have $\varphi_t(g) \to 1$ as $t \to 0$ for all $g \in \mathbb{F}_n$. Further, $\|b(g)\| \to \infty$ if $g \to \infty$ implies that for all $t \in (0, \infty)$ we have $\varphi_t(g) \to 0$ as $g \to \infty$. This shows that $\mathbb{F}_n$ has the Haagerup property. ☐

Proposition 3.3.3. *Let $\Gamma \curvearrowright X$ be a pmp action and let $\varphi_i : \Gamma \to \mathbb{C}$ be sequence of normalised positive type functions converging to 1 pointwise. The normal ucp maps $\mathrm{L}^\infty(X)$-bimodular maps $\Phi_i : \mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(X) \rtimes \Gamma$ satisfying $\Phi_i(u_g) = \varphi_i(g)u_g$ converge to* id *pointwise in $\|\ \|_2$.*

Proof. Let $x = \sum_{g\in\Gamma} x_g u_g$ an element in the unit ball of $\mathrm{L}^\infty(X) \rtimes \Gamma$. Then $\Phi_i(x) = \sum_{g\in\Gamma} \varphi_i(g)x_g u_g$. Let $\varepsilon > 0$ and take $F \subset \Gamma$ finite such that $\sum_{g\in\Gamma\setminus F} \|x_g\|_2^2 < \varepsilon$. For all i satisfying $|\varphi_i(g) - 1|^2 \leq \varepsilon/|F|$ for all $g \in F$, we have the estimate

$$\begin{aligned}\|\Phi_i(x) - x\|_2^2 &= \sum_{g\in\Gamma} \|(\varphi_i(g) - 1)x_g\|_2^2 \\ &= \sum_{g\in F} \underbrace{\|(\varphi_i(g) - 1)x_g\|_2^2}_{\leq \|x\|^2\varepsilon/|F|} + \sum_{g\in\Gamma\setminus F} \underbrace{(\varphi_i(g) - 1)}_{\leq 2}\|x_g\|_2^2 \\ &\leq |F|\|x\|^2\varepsilon/|F| + 2\varepsilon \\ &= (\|x\|^2 + 2)\varepsilon\end{aligned}$$

Since $\varphi_i \to 1$ pointwise, this shows that $\|\Phi_i(x) - x\|_2 \to 0$, finishing the proof of the proposition. ☐

3.4 Rigid inclusions of von Neumann algebras and rigid actions

We are going to exploit the conclusion of Proposition 3.3.3 by opposing it with the following rigidity property for inclusions of von Neumann algebras.

Definition 3.4.1. Let $A \subset M$ be a tracial inclusion of von Neumann algebras. Then $A \subset M$ is called rigid if for all sequences of normal tracial ucp maps $\Phi_i : M \to M$ that converge to id_M pointwise in $\| \ \|_2$, we have $\Phi_i \to \mathrm{id}$ uniformly in $\| \ \|_2$ on the unit ball $(A)_1$.

Our source of rigid inclusions of von Neumann algebras is the relative property (T) for inclusions of groups.

Definition 3.4.2. Let $\Lambda \leq \Gamma$ be an inclusion of groups. Then $\Lambda \leq \Gamma$ has relative property (T), if every sequence of positive type functions $\varphi_i : \Gamma \to \mathbb{C}$ converging pointwise to 1 converges uniformly on Λ.

Proposition 3.4.3. *Let $\Lambda \leq \Gamma$ be an inclusion of groups with relative property (T). Then $\mathrm{L}(\Lambda) \subset \mathrm{L}(\Gamma)$ is rigid.*

Proof. Let $\Phi_i : \mathrm{L}(\Gamma) \to \mathrm{L}(\Gamma)$ be a sequence of normal tracial ucp maps converging to id pointwise. Let $\varphi_i(g) \coloneqq \tau(\Phi(u_g)u_g^*)$. Then $\varphi_i \to 1$ pointwise. For $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $g_1, \dots, g_n \in \Lambda$ we have

$$
\begin{aligned}
\sum_{\alpha,\beta=1}^{n} \lambda_\alpha \overline{\lambda_\beta} \varphi_i(g_\beta^{-1} g_\alpha) &= \sum_{\alpha,\beta=1}^{n} \lambda_\alpha \overline{\lambda_\beta} \tau(\Phi_i(u_{g_\beta}^* u_{g_\alpha}) u_{g_\alpha}^* u_{g_\beta}) \\
&= \sum_{\alpha,\beta=1}^{n} \lambda_\alpha \overline{\lambda_\beta} \langle u_{g_\beta}^* u_{g_\alpha} \otimes u_{g_\alpha}^* u_{g_\beta}, \xi_{\Phi_i} \rangle \\
&= \sum_{\alpha,\beta=1}^{n} \langle \lambda_\alpha u_{g_\alpha} \otimes u_{g_\alpha}^*, \lambda_\beta u_{g_\beta} \otimes u_{g_\beta}^* \rangle \\
&\geq 0 \,,
\end{aligned}
$$

where we made use of the $\mathrm{L}(\Gamma)$-$\mathrm{L}(\Gamma)$-bimodule $\mathcal{H}_{\Phi_i}$. We showed that φ_i is a positive type function. Since $\Lambda \leq \Gamma$ has relative property (T), $\varphi_i \to 1$ uniformly on Λ. Let $\varepsilon > 0$ and take i such that $|\varphi_i(g) - 1| < \varepsilon$ for all $g \in \Lambda$. Write $\mathcal{H}_i = \mathcal{H}_{\Phi_i}$ with generating vector $\xi_i = \xi_{\Phi_i}$. We have

$$
|\langle u_g \xi_i u_g^*, \xi_i \rangle - 1| = |\varphi_i(g) - 1| < \varepsilon \,.
$$

So $\| u_g \xi_i u_g^* - \xi \|^2 = 2 - 2 \mathfrak{Re} \langle u_g \xi_i u_g^*, \xi_i \rangle < 2\varepsilon$. Let η_i be the barycentre of the convex set $\overline{\mathrm{conv}} \{ u_g \xi_i u_g^* \,|\, g \in \Lambda \}$. Then $\eta_i = u_g \eta_i u_g^*$ for all $g \in \Lambda$, implying that $x \eta_i = \eta_i x$ for all $x \in \mathrm{L}(\Lambda)$. Further $\|\eta_i - \xi_i\|^2 < 2\varepsilon$. If $u \in \mathcal{U}(\mathrm{L}(\Lambda))$, then

$$
\begin{aligned}
\|\Phi_i(u) - u\|_2^2 &= 2 - 2 \mathfrak{Re} \langle \Phi_i(u), u \rangle \\
&= 2 - 2 \mathfrak{Re} \langle u \xi_i u^*, \xi \rangle \\
&= \| u \xi_i - \xi_i u \|_2^2 \\
&= \| u(\xi_i - \eta_i) - (\xi_i - \eta_i) u \|_2^2 \\
&\leq (2 \|\xi_i - \eta_i\|_2)^2 \\
&\leq 8\varepsilon \,.
\end{aligned}
$$

Since every element in $(\mathrm{L}(\Lambda))_1$ is a sum of 4 unitaries, it follows that $(\Phi_i)_i$ converges to id uniformly on the unit ball of $\mathrm{L}(\Lambda)$. ☐

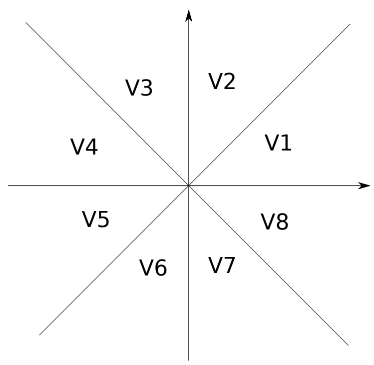
Theorem 3.4.4. *The inclusion $\mathbb{Z}^2 \leq \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ has relative property (T).*

Lemma 3.4.5. *Let ν be a measure on $\mathbb{R}^2$ which is supported on $[-1/2,1/2]^2$ and which is ε-invariant under the transformations*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $\nu(0) \geq 1 - 8\varepsilon$.

Proof. Denote by $V_1, V_2, \dots, V_8$ the eight parts of the real plane between the coordinate axes and the lines $x = x$ and $x = -x$. We consider the ray which is on the positively oriented side of these parts as belonging to them. The origin is not part of either of $V_1, V_2, \dots, V_8$. This gives us the following partition of $\mathbb{R}^2 \setminus \{0\}$.



We show that $\nu(V_i) \leq \varepsilon$ for all i, so that $\nu(\{0\}) = 1 - \sum_{i=1}^{8} \nu(V_i) \geq 1 - 8\varepsilon$ will follow.

Denote by α the linear transformation of $\mathbb{R}^2$ induced by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $\alpha^{-1}(V_1) = V_1 \cup V_2$. Since $\|\nu - \nu \circ \alpha\|_1 \leq \varepsilon$, it follows that $\nu(V_2) \leq \varepsilon$. Since $\alpha^{-1}(V_8) = V_7 \cup V_8$, we obtain $\nu(V_7) \leq \varepsilon$, too. Using ε-invariance of ν under α^{-1}, we obtain likewise that $\nu(V_3), \nu(V_6) \leq \varepsilon$. Considering the same argument with the linear transformation induced by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

we see that $\nu(V_1), \nu(V_4), \nu(V_5), \nu(V_8) \leq \varepsilon$. This finishes the proof of the lemma ☐

Proof of Theorem 3.4.4. Write $\Gamma = \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$. Let $\varphi_i : \Gamma \to \mathbb{C}$ be a sequence of positive type functions converging to 1 pointwise. By Theorem 3.2.2 there is a the cyclic representation representation of Γ on a Hilbert space H_i with cyclic vector ξ_i such that $\varphi_i(g) = \langle g\xi_i, \xi_i \rangle$ for all $g \in \Gamma$. Using the isomorphism $\mathrm{C}^*(\mathbb{Z}^2) \cong \mathrm{C}(\hat{\mathbb{Z}}^2) \cong \mathrm{C}(\mathbb{T}^2)$, we obtain a representation of $\mathrm{C}(\mathbb{T}^2)$ on H_i, which is equivariant with respect to the $\mathrm{SL}(2,\mathbb{Z})$-action. The vector state $f \mapsto \langle f\xi_i, \xi_i \rangle$ on $\mathrm{C}(\mathbb{T}^2)$ defines a measure ν_i on $\mathbb{T}^2$. We identify $\mathbb{T} \subset \mathbb{C}$ with the elements of length 1 and hence $\mathbb{T}^2 \subset \mathbb{C}^2$. For all $m, n \in \mathbb{Z}$, we have

$$\int_{\mathbb{T}^2} z_1^n z_2^m \mathrm{d}\nu_i(z_1, z_2) = \varphi(n, m) \to 1,$$

showing by the Stone-Weierstrass theorem that $\int_{\mathbb{T}^2} f \mathrm{d}\nu_i \to f(1,1)$ for all $f \in \mathrm{C}(\mathbb{T}^2)$. In particular, for $B := \{(\mathrm{e}^{2\pi i t_1}, \mathrm{e}^{2\pi i t_2}) \mid t_1, t_2 \in [-1/4, 1/4]\}$ we have $\nu_i(B) \to 1$. If $g \in \Gamma$, then

$$\begin{aligned}
\Big|\int_{\mathbb{T}^2} {}^g f \mathrm{d}\nu_i - \int_{\mathbb{T}^2} f \mathrm{d}\nu_i\Big| &= |\langle u_g f u_g^* \xi_i, \xi_i\rangle - \langle f\xi_i, \xi_i\rangle| \\
&= |\langle f u_g^* \xi_i, u_g^* \xi_i\rangle - \langle f\xi_i, \xi_i\rangle + \langle f\xi_i, u_g^*\xi_i\rangle - \langle f\xi_i, u_g^*\xi_i\rangle| \\
&\leq |\langle f u_g^* \xi_i - f\xi_i, u_g^*\xi_i\rangle| + |\langle f\xi_i, \xi_i - u_g^*\xi_i\rangle| \\
&\leq \|f u_g^*\xi_i - f\xi_i\| \|u_g^*\xi_i\| + \|f\xi_i\| \|\xi_i - u_g^*\xi_i\| \\
&\leq 2\|f\| \|u_g^*\xi_i - \xi_i\| .
\end{aligned}$$

This shows that $\|g\nu_i - \nu_i\|_1 \to 0$ for all $g \in \Gamma$. Let $0 < \varepsilon$. Take i such that $\nu_i(B) \geq 1-\varepsilon$ and $\|g\nu_i - \nu_i\|_1 < \varepsilon$ for all $g \in F$, where

$$F = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right\} \subset \mathrm{SL}(2, \mathbb{Z}) .$$

Let $\tilde{\nu}_i := \nu_i|_B / \nu_i(B)$. Then

$$\|\tilde{\nu}_i - \nu_i\|_1 = 2(1 - \nu_i(B)) \leq 2\varepsilon$$

and

$$\|g\tilde{\nu}_i - \tilde{\nu}_i\|_1 \leq 5\varepsilon ,$$

for all $g \in F$. Since $g[-1/4, 1/4]^2 \subset [-1/2, 1/2]^2$ for all $g \in F$, we can consider $\tilde{\nu}_i$ as measure on $\mathbb{R}^2$ supported in $[-1/4, 1/4]^2$, which is 5ε-invariant under the linear transformation induced by elements from F. Now Lemma 3.4.5 implies that $\tilde{\nu}_i((0,0)) \geq 1 - 40\varepsilon$. Since $\|\tilde{\nu}_i - \nu_i\| \leq 2\varepsilon$, this shows that $\nu_i((1,1)) \geq 1 - 42\varepsilon$. Hence, $\|\nu_i - \delta_{(1,1)}\|_1 \leq 2(1 - \nu_i((1,1))) \leq 84\varepsilon$. It follows that

$$|\varphi_i(m,n) - 1| = \Big|\int_{\mathbb{T}^2} z_1^m z_2^n \mathrm{d}\nu_i(z_1, z_2) - \int_{\mathbb{T}^2} z_1^m z_2^n \mathrm{d}\delta_{(1,1)}\Big| \leq \|\nu_i - \delta_{(1,1)}\|_1 \sup_{z_1, z_2 \in \mathbb{T}} |z_1^m z_2^n| \leq 84\varepsilon .$$

This proves that $\varphi_i \to 1$ uniformly on $\mathbb{Z}^2$ as $i \to \infty$. ☐

3.5 A uniqueness of Cartan result

Theorem 3.5.1. *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be free ergodic pmp actions and assume that*

- *Γ has the Haagerup property and*
- *$\mathrm{L}^\infty(Y) \subset \mathrm{L}^\infty(Y) \rtimes \Lambda$ is rigid.*

If $\alpha : \mathrm{L}^\infty(X) \rtimes \Gamma \xrightarrow{\cong} \mathrm{L}^\infty(Y) \rtimes \Lambda$, then there is a unitary $u \in \mathcal{U}(\mathrm{L}^\infty(Y) \rtimes \Lambda$ such that $u\alpha(\mathrm{L}^\infty(X))u^* = \mathrm{L}^\infty(Y)$. In particular, $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are orbit equivalent.

Proof. We identify $\mathrm{L}^\infty(X) \rtimes \Gamma = \mathrm{L}^\infty(Y) \rtimes \Lambda = M$ via the isomorphism α. Write $A = \mathrm{L}^\infty(Y)$, $B = \mathrm{L}^\infty(X)$. Denote by $\varphi_i : \Gamma \to \mathbb{C}$ a sequence of normalised positive type functions in $\mathrm{c}_0(\Gamma)$ such that $\varphi_i \to 1$ pointwise. By Proposition 3.2.4 there are normal B-modular trace preserving ucp maps $\Phi_i : M \to M$ such that $\Phi_i(u_g) = \varphi_i(g) u_g$ for all $g \in \Gamma$. Proposition 3.3.3 says that $\Phi_i \to \mathrm{id}_M$ pointwise in $\| \ \|_2$. Since $\Lambda \curvearrowright Y$ is rigid, also $A \subset M$ is rigid by Theorem 3.4.3. So $\Phi_i \to \mathrm{id}$ uniformly in $\| \ \|_2$ on the unit ball of A.

Assume that there is a sequence of unitaries $(u_n)_n$ is A such that $\mathrm{E}_B(xu_ny) \to 0$ in $\| \ \|_2$ for all $x, y \in M$. Then in particular $(u_n)_g = \mathrm{E}_B(u_n u_g^*) \to 0$ in $\| \ \|_2$ for all $g \in \Gamma$. Let $0 < \varepsilon < 1/5$ and take i such that $\|\Phi_i(u_n) - u_n\|_2 < \varepsilon$ for all $n \in \mathbb{N}$. Let $F \subset \Gamma$ be a finite subset such that $\varphi_i(g) < \varepsilon$ for all $g \in \Gamma \setminus F$. Now take n such that $\|(u_n)_g\|_2^2 < \varepsilon/|F|$ for all $g \in F$. Then

$$\begin{aligned}
1 &= \|u_n\|_2^2 \\
&\leq (\|u_n - \Phi_i(u_n)\|_2 + \|\Phi_i(u_n)\|_2)^2 \\
&\leq \|\Phi_i(u_n)\|_2^2 + (\varepsilon^2 + 2\varepsilon) \\
&= \|\sum_{g\in\Gamma} (u_n)_g \Phi_i(u_g)\|_2^2 + (\varepsilon^2 + 2\varepsilon) && B\text{-modularity} \\
&= \|\sum_{g\in\Gamma} \varphi_i(g)(u_n)_g u_g\|_2^2 + (\varepsilon^2 + 2\varepsilon) \\
&= \sum_{g\in F} \underbrace{\|\varphi_i(g)(u_n)_g\|_2^2}_{\leq |\varphi_i(g)|^2 \varepsilon / F} + \sum_{g\in\Gamma\setminus F} \underbrace{\|\varphi_i(g)(u_n)_g\|^2}_{\leq \varepsilon^2 \|(u_n)_g\|_2^2} + (\varepsilon^2 + 2\varepsilon) \\
&\leq \varepsilon + \varepsilon^2 + \varepsilon^2 + 2\varepsilon \\
&< 1 .
\end{aligned}$$

This is a contradiction. So Theorem 2.6.1 shows that $A \prec_M B$, which implies by Theorem 2.6.3 that there is a unitary $u \in \mathcal{U}(M)$ such that $uAu^* = B$. In particular, $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are orbit equivalent by Theorem 1.4.37. □

3.6 Cost

In view of Theorem 3.5.1, we now want to introduce orbit equivalence invariants for group actions. Gaboriau introduced an invariant, generalising the rank of a group. We will introduce this invariant — cost — and then prove that the cost of every free pmp action of a free group equals the rank of the free group.

Definition 3.6.1. Let (X, μ) be a standard probability measure space without atoms. A countable probability measure preserving equivalence relation is an equivalence relation $\mathcal{R}$ on X such that $\mathcal{R} \subset X \times X$ is measurable, $\mathcal{R}$ has countable classes and such that for every partial isomorphism $\varphi : A \to B$ with $A, B \subset X$ and $\operatorname{graph} \varphi \subset \mathcal{R}$ we have $\mu(A) = \mu(B)$. A countable probability measure preserving equivalence relation is called II_1 equivalence relation if it has almost surely infinite classes.

Two countable pmp equivalence relations $\mathcal{R} \subset X \times X$ and $\mathcal{S} \subset Y \times Y$ are called isomorphic if there is an isomorphism $\Delta : X \to Y$ such that $(\Delta \times \Delta)(\mathcal{R}) = \mathcal{S}$.

Example 3.6.2. Let $\Gamma \overset{\alpha}{\curvearrowright} X$ be a free pmp action of an infinite discrete group. Then $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X) = \{(x, gx) \mid x \in X, g \in \Gamma\}$ is a II_1 equivalence relation.

Indeed, $\mathcal{R}$ is an equivalence relation, since Γ is a group. Further, $\mathcal{R} = \bigcup_{g\in\Gamma} \operatorname{graph} \alpha_g$ is measurable as a subset of $X \times X$. Since Γ is countable, it is immediate that $\mathcal{R}$ has countable classes. These classes are almost surely infinite, since Γ acts freely. Finally if $\varphi : A \to B$ is a partial isomorphism between $A, B \subset X$ such that $\operatorname{graph} \varphi \subset \mathcal{R}$, then $A = \bigsqcup_{g\in\Gamma} A_g$ with $\varphi|_{A_g} = \alpha_g|_{A_g}$, so that $\mu(A) = \mu(B)$ follows.

The next proposition is obvious from the definitions.

Proposition 3.6.3. *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be free pmp actions. Then $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$ if and only if $\Gamma \curvearrowright X \sim_{OE} \Lambda \curvearrowright Y$.*

Definition 3.6.4. Let $\mathcal{R}$ be a II_1 equivalence relation. A graphing of $\mathcal{R}$ is a family Φ of maps $\varphi : A_\varphi \to B_\varphi$ of partial isomorphism with $\mathrm{graph}(\varphi) \subset \mathcal{R}$ such that $\mathcal{R}$ is generated by $\bigcup_{\varphi \in \Phi} \mathrm{graph}\,\varphi$.

We define the cost of a graphing Φ as $\mathcal{C}(\Phi) = \sum_{\varphi \in \Phi} A_\varphi$ and the cost of the equivalence relation $\mathcal{R}$ as $\mathcal{C}(\mathcal{R}) = \inf_{\Phi \text{ graphing of } \mathcal{R}} \mathcal{C}(\Phi)$.

Let us remark that cost is invariant under isomorphism of II_1 equivalence relations. Further, it is a generalisation of the rank of a group. In fact, if $\Gamma = \langle S \rangle$ is a finitely generated group and $\Gamma \overset{\alpha}{\curvearrowright} X$ is a free pmp action, then $\Phi = \alpha_g)_{g \in S}$ is a graphing of $\mathcal{R}(\Gamma \curvearrowright X)$ with cost $|S|$.

The following is a special case of a result of Gaboriau.

Theorem 3.6.5. *Let $\mathbb{F}_n \curvearrowright X$ be a free pmp action and $\mathcal{R} = \mathcal{R}(\mathbb{F}_n \curvearrowright X)$. Then $\mathcal{C}(\mathcal{R}) = n$.*

Proof. We already remarked that $\mathcal{C}(\mathcal{R}) \leq n$, so we have to show the converse inclusion. Let $S \subset \mathbb{F}_n$ be its natural set of generators and denote by μ the probabilty measure on X. Cutting and pasting we may assume that every graphing is of the form $(\varphi_g : A_g \to B_g)_{g \in \mathbb{F}_n}$ with $\varphi_g = \alpha_g|_{A_g}$. Let $\varepsilon > 0$. We define a partial ordering on graphings Φ of $\mathcal{R}$ satisfying $\mathcal{C}(\Phi) \leq \mathcal{C}(\mathcal{R}) + \varepsilon$ by

$$\Phi \leq \Phi' \quad \text{if and only if} \quad \forall g \in S : A_g \subset A'_g \text{ and } \forall g \in \mathbb{F}_n \setminus S : A_g \supset A'_g .$$

We may find a maximal element Φ for this order and we shall prove that $A_g = X$ for all $g \in S$. This will finish the proof of the theorem.

So assume from now on for a contradiction, that $A_h \neq X$ for some $h \in S$. Since Φ generates $\mathcal{R}$, we find partial isomorphisms $\varphi_i : A_i \to B_i = A_{i+1}$, $i \in \{1, \dots, l\}$ such that

- $A_1 \subset A$,
- there are elements $g_i \in \mathbb{F}_n$ such that φ_i is the restriction of φ_{g_i} for all i,
- $\varphi_l \circ \varphi_{l-1} \circ \cdots \circ \varphi_1 = \alpha_h|_{A_1}$.

Since $\mathbb{F}_n \curvearrowright X$ is free, it follows that $g_l \cdots g_1 = h$. Hence there is $k \in \{1, \dots, l\}$ and there are $g_{k,1}, g_{k,2} \in \mathbb{F}_n$ such that

- $g_k = g_{k,1} h g_{k,2}$,
- $g_l g_{l-1} \cdots g_{k+1} g_{k,1} = e$, and
- $g_{k,2} g_{k-1} \cdots g_1 = e$.

First assume that $g_k \in S$ is a generator. Then $g_k = h$. Further, $g_{k-1} \cdots g_1 = e$, so it follows that $A_1 = A_k \subset A_h$, in contradiction to the assumption $A_1 \subset A = X \setminus A_h$. We showed that $g_k \notin S$. Since $g_l g_{l-1} \cdots g_{k+1} = g_{k,1}^{-1}$ and $g_{k-1} \cdots g_1 = g_{k,2}^{-1}$, we may remove then $\varphi_k = \varphi_{g_k}|_{A_K}$ from Φ and add instead $\alpha_h|_{g_{k,2} g_{k-1} \cdots g_1 A_1}$ to Φ and we obtain a graphing Φ' of $\mathcal{R}$. Since $g_{k,2} g_{k-1} \dots g_1 = e$, we have $\alpha_h|_{g_{k,2} g_{k-1} \cdots g_1 A_1} = \alpha_h|_{A_1}$, so that $\Phi < \Phi'$. This contradicts the maximality of Φ and finishes the proof of the theorem. ☐

3.7 Some examples of non-isomorphic group measure space constructions

In this section we are going to exploit Theorem 3.5.1 together with Theorem 3.6.5. We are going to construct rigid free ergodic pmp actions of free groups of arbitrary rank. Continuing our example from Section 3.4, we will construct these as restrictions of $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2$ to certain finite index free subgroups.

We will make use of the following proposition, for which we do not give a proof.

Proposition 3.7.1. *Let $\Gamma \curvearrowright X$ be a rigid free ergodic pmp action. If $\Lambda \leq \Gamma$ is a finite index subgroup, then also $\Lambda \curvearrowright X$ is rigid.*

Let us exhibit free subgroups of arbitrary rank in $\mathrm{SL}(2,\mathbb{Z})$.

Proposition 3.7.2. *Let Γ be the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by the matrices*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Then $\Gamma \cong \mathbb{F}_2$.

In order to give a proof of this proposition, we will make use of the so called ping-pong lemma. This is a very useful result, which appears in different forms in group theory.

Lemma 3.7.3. *Let Γ be a group generated by subgroups $\Gamma_1, \dots, \Gamma_n \leq \Gamma$, such that the product of their orders is at least 6. Assume that there is an action $\Gamma \curvearrowright X$ on a set such that there are subsets $X_1, \dots, X_n \subset X$ satisfying $(\Gamma_i \setminus \{e\})X_j \subset X_i$ for all $i \neq j$. Then $\Gamma = \Gamma_1 * \dots * \Gamma_n$.*

Proof. We may assume that all $\Gamma_1, \dots, \Gamma_n$ are non-trivial. If $n = 1$, we are done. If $n = 2$, then either Γ_1 or Γ_2 has order at least three and we may assume that this is Γ_1. We will show that every non-trivial reduced word in the elements from $\Gamma_1, \dots, \Gamma_n$ acts non-trivially on X. This will show that the natural homomorphism $\Gamma_1 * \dots * \Gamma_n \to \Gamma$ is injective. Since it is surjective by assumption, this will finish the proof.

Let w be a non-trivial reduced word in the elements of $\Gamma_1, \dots, \Gamma_n$. Let us first assume that the first and the last letter of w are from the same Γ_i. Then $wX_j \subset X_i$ for $i \neq j$, by our assumptions. So $w \neq e$ in Γ. Now let w be arbitrary. If $n \geq 3$ and the first and the last letter of w are from Γ_i and $\Gamma_{i'}$ respectively, we consider $wX_j \subset X_i$ for some $j \neq i, i'$. Again, we obtain $w \neq e$ in Γ. If $n = 2$, we can conjugate w with some element from Γ_1 in order to obtain a word which does start and end with an element from Γ_1. To this end recall that $|\Gamma_1| \geq 3$. We can hence find an element $g \in \Gamma_1$ which neither equals the inverse of the first element of w nor equals the last element of w. (Only one of these two lies in Γ_1). We obtain by the previous argument that $gwg^{-1} \neq e$ in Γ, so that $w \neq e$ in Γ. This finishes the proof of the lemma. □

Proof of Proposition 3.7.2. We apply the Ping-Pong Lemma to the action $\Gamma \leq \mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{R}^2$. Let

$$\Gamma_1 = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \rangle \quad \Gamma_2 = \langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle.$$

Further let

$$X_1 = \{(x,y)^{\mathrm{t}} \in \mathbb{R}^2 \mid |x| > |y|\} \quad X_2 = \{(x,y)^{\mathrm{t}} \in \mathbb{R}^2 \mid |y| > |x|\}.$$

Then $(\Gamma_1 \setminus \{e\})X_2 \subset X_1$ and $(\Gamma_2 \setminus \{e\})X_1 \subset X_2$. Since Γ_1, Γ_2 are both infinite cyclic groups, it follows from the Ping-Pong Lemma 3.7.3 that $\Gamma = \Gamma_1 * \Gamma_2 \cong \mathbb{F}_2$. □

We want to determine the index of Γ from Proposition 3.7.2 inside $\mathrm{SL}(2,\mathbb{Z})$.

Proposition 3.7.4. *Let Γ be the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by the matrices*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Then $\Gamma \leq \mathrm{SL}(2,\mathbb{Z})$ has index 12.

Proof. Consider the so called congruence subgroup $\Gamma_{(2)} := \ker(\mathrm{SL}(2,\mathbb{Z}) \twoheadrightarrow \mathrm{SL}(\mathbb{Z}/2\mathbb{Z}))$. Since $\mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$ has order 6, it follows that $[\Gamma : \Gamma_{(2)}] = 6$. We will show that $\Gamma_{(2)}$ is generated by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It then follows that $[\Gamma_{(2)} : \Gamma] = 2$, implying the proposition.

Every element in $\Gamma_{(2)}$ must be of the form

$$g = \begin{pmatrix} 1+2k & 2l \\ 2m & 1+2n \end{pmatrix}$$

for some $k, l, m, n \in \mathbb{Z}$. If $l = 0$ or $m = 0$ then it is clear that $g \in \Gamma_{(2)}$. Otherwise, we show that we can multiply from the left with matrices

$$\begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}$$

in order to reduce the absolute value of the off-diagonal entries. We have

$$\begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+2k & 2l \\ 2m & 1+2n \end{pmatrix} = \begin{pmatrix} 1+2(k\pm 2m) & 2(l\pm(1+2n)) \\ 2m & 1+2n \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}\begin{pmatrix} 1+2k & 2l \\ 2m & 1+2n \end{pmatrix} = \begin{pmatrix} 1+2k & 2l \\ 2(m\pm(1+2k) & 1+2(n\pm 2l) \end{pmatrix}.$$

So it is indeed possible to reduce the absolute value of one of the off-diagonal entries, unless $|1+2n| \geq |2l|$ and $|1+2k| \geq |2m|$. But this cannot happen as the following calculations show. If $|1+2n| = |2l|$ and $|1+2k| = |2m|$, then

$$\det\begin{pmatrix} 1+2k & 2l \\ 2m & 1+2n \end{pmatrix} = (1+2k)(1+2n) - 4lm \in \{0, 8lm\}.$$

If one of the inequalities $|1+2n| \geq |2l|$ and $|1+2k| \geq |2m|$ is proper, we obtain

$$\begin{aligned} \left|\det\begin{pmatrix} 1+2k & 2l \\ 2m & 1+2n \end{pmatrix}\right| &\geq |1+2k||1+2n| - |4lm| \\ &\geq \min\{(|2l|+1)|2m|, |2l|(|2m|+1)\} - |4lm| \\ &= \min\{|2m|, |2l|\}. \end{aligned}$$

Since $l, m \neq 0$, this is a contradiction, finishing the proof. ☐

We showed that $\mathrm{SL}(2,\mathbb{Z})$ has a finite index subgroup isomorphic with $\mathbb{F}_2$. The following theorem of Nielsen-Schreier, which we will not prove, then provides finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ or arbitrary rank.

Theorem 3.7.5 (Nielsen-Schreier)**.** *Let $\Gamma \leq \mathbb{F}_n$ be some subgroup of index k. Then $\Gamma \cong \mathbb{F}_m$ for $m = (n-1)k+1$.*

Note that the kernel of $\mathbb{F}_2 \to \mathbb{Z}/k\mathbb{Z}$, sending the first generator to 1 and the second generator to 0 has index k inside $\mathbb{F}_2$. So $\mathbb{F}_2$ contains finite index free groups of rank $(2-1)k+1 = k+1$ for every $k \geq 2$.

Summarising our results up to now, we showed the following statement: for every $n \in \mathbb{N}_{\geq 2}$ there is a rigid free pmp action of $\mathbb{F}_n \curvearrowright \mathbb{T}^2$ coming from the restiction of $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2$. Note that if $\Lambda \leq \Gamma$ is a finite index inclusion and $\Gamma \curvearrowright X$ is an ergodic pmp action, then any Λ-invariant set must have measure at least $[\Gamma : \Lambda]^{-1}$. We can hence find for every $n \in \mathbb{N}_{\geq 2}$ some rigid freeergodic pmp action of $\mathbb{F}_n$. We can now combine Theorems 3.6.5 and 3.5.1 with our work in this section to obtain the following conclusion.

Theorem 3.7.6. *There are free ergodic pmp actions $\mathbb{F}_n \curvearrowright X$ of non-abelian free groups of arbitrary rank such that $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n \cong \mathrm{L}^\infty(X) \rtimes \mathbb{F}_m$ implies $n = m$.*